

Consistent estimation of the filtering and marginal smoothing distributions in nonparametric hidden Markov models

Yohann De Castro*

Élisabeth Gassiat*

Sylvain Le Corff*

July 24, 2015

Abstract

In this paper, we consider the filtering and smoothing recursions in nonparametric finite state space hidden Markov models (HMMs) when the parameters of the model are unknown and replaced by estimators. We provide an explicit and time uniform control of the filtering and smoothing errors in total variation norm as a function of the parameter estimation errors. We prove that the risk for the filtering and smoothing errors may be uniformly upper bounded by the risk of the estimators. It has been proved very recently that statistical inference for finite state space nonparametric HMMs is possible. We study how the recent spectral methods developed in the parametric setting may be extended to the nonparametric framework and we give explicit upper bounds for the L^2 -risk of the nonparametric spectral estimators. When the observation space is compact, this provides explicit rates for the filtering and smoothing errors in total variation norm. The performance of the spectral method is assessed with simulated data for both the estimation of the (nonparametric) conditional distribution of the observations and the estimation of the marginal smoothing distributions.

1 Introduction

1.1 Context and motivations

Hidden Markov models are popular time evolving models to depict practical situations in a variety of applications such as economics, genomics, signal processing and image analysis, ecology, environment, speech recognition, see [11] for a recent overview of HMMs. Finite state space HMMs are stochastic processes $(X_j, Y_j)_{j \geq 1}$ such that $(X_j)_{j \geq 1}$ is a Markov chain with finite state space \mathcal{X} and $(Y_j)_{j \geq 1}$ are random variables with general state space \mathcal{Y} , independent conditionally on $(X_j)_{j \geq 1}$ and such that for all $\ell \geq 1$, the conditional distribution of Y_ℓ given $(X_j)_{j \geq 1}$ depends on X_ℓ only. The observations are $Y_{1:n} := (Y_1, \dots, Y_n)$ and the associated states $X_{1:n} := (X_1, \dots, X_n)$ are unobserved. The parameters of the model are the initial distribution π^* of the hidden chain, the transition matrix of the hidden chain \mathbf{Q}_* and the conditional distribution of Y_1 given $X_1 = x$ for all possible $x \in \mathcal{X}$ which are often called emission distributions.

In many applications of finite state space HMMs (e.g. digital communication or speech recognition), it is of utmost importance to infer the sequence of hidden states. Such inference usually involves the computation of the posterior distribution of a set of hidden states $X_{k:k'}, 1 \leq k \leq k' \leq n$, given the observations $Y_{1:s}, 1 \leq s \leq n$. When the initial distribution of the hidden chain, its transition matrix and the conditional distribution of the observations are known, this task can be efficiently done using the forward-backward algorithm described in [5] and [25]. In this paper, we focus on the estimation of the filtering distributions $\mathbb{P}(X_k = x | Y_{1:k})$ and marginal smoothing distributions $\mathbb{P}(X_k = x | Y_{1:n})$ for all $1 < k \leq n$ when the parameters of the HMM are unknown and replaced by estimators. Indeed, it has been proved very recently that inference in finite state space nonparametric HMMs is possible, see [16].

1.2 Contribution

The aim of our paper is twofold.

- First, we study how the parameter estimation error propagates to the error made on the estimation of filtering and smoothing distributions. Although replacing parameters by their estimators to compute posterior distributions and infer the hidden states is usual in applications, theoretical results to support this practice are

¹Laboratoire de Mathématiques d'Orsay (CNRS UMR 8628), Université Paris-Sud 11, F-91405 Orsay Cedex, France.

very few regarding the accuracy of the estimated posterior distributions. We are only aware of [15] whose results are restricted to the filtering distribution in a parametric setting. When the parameters of the HMM are known, the forward-backward algorithm can be extended to general state space HMMs or when the cardinality of \mathcal{X} is too large using computational methods such as Sequential Monte Carlo methods (SMC), see [8, 12] for a review of these methods. In this context, the Forward Filtering Backward Smoothing [21, 19, 13] and Forward Filtering Backward Simulation [17] algorithms have been intensively studied, with the objective of quantifying the error made when the filtering and marginal smoothing distributions are replaced by their Monte Carlo approximations. These algorithms and some extensions have been analyzed theoretically recently, see for instance [9, 10, 14, 23]. SMC methods may also be used in algorithms when the parameters of the HMM are unknown to perform maximum likelihood parameter estimation, see [20] for on-line and off-line Expectation Maximization and gradient ascent based algorithms. Part of our analysis of the filtering and smoothing distributions is based on the same approach as in those papers and requires sharp forgetting properties of HMMs.

- Second, we extend spectral methods to a nonparametric setting and give explicit control of the L^2 -risk of the estimators. Such estimators may then be used in the computation of posterior distributions. In latent variable models such as HMMs, spectral methods are popular since they lead to algorithms that are not sensitive to a chosen initial estimate. Indeed, standard estimation methods for HMMs are based on the Expectation-Maximization (EM) algorithm, which faces intrinsic limitations hard to circumvent such as slow convergence and suboptimal local optima. Extending spectral methods to nonparametric HMMs is thus very useful. In particular, they may be used to provide a preliminary estimator as starting point in a EM algorithm. They are also used in a refinement procedure proposed in [7]. To the best of our knowledge, the spectral method has not been extended nor studied yet in the nonparametric framework.

We start from the works of Anandkumar, Hsu, Kakade and Zhang on spectral methods in the parametric frame. Their papers [18, 3] present an efficient algorithm for learning parametric HMMs or more generally finitely many linear functionals of the parameters of the HMM. Thus, it is possible to use spectral methods to estimate the projections of the emission distributions onto nested subspaces of increasing complexity. Our work brings a new quantitative insight on the tradeoff between sampling size and approximation complexity for spectral estimators. We provide a nonasymptotic precise upper bound of the risk for the variance term with respect to the number of observations and the complexity of the approximating subspace.

1.3 Outline of the paper

In section 2, we provide an explicit control of the total variation filtering and smoothing errors as a function of the parameter estimation error, see Propositions 2.1 and 2.2. We detail the application of these preliminary results in the parametric context, see Theorem 2.3, and in the nonparametric context, see Theorem 2.4 where we prove that the uniform rate of convergence for the filtering and smoothing errors is driven by the L^1 -risk of the nonparametric estimator of the emission distributions. In Section 3, we explain how spectral methods can be extended to the nonparametric frame and we provide the nonasymptotic control of the variance term in Theorem 3.1. This leads to the asymptotic behavior proved in Corollary 3.2, which may be invoked when spectral methods are used in the computation of posterior distributions, see Corollary 2.5 stated in the previous section. Finally, in Section 4 we show the performance of the spectral method with simulated data for both the estimation of the (nonparametric) conditional distribution of the observations and the estimation of the marginal smoothing distributions. All detailed proofs are given in the appendices.

2 Main results

2.1 Notations and setting

In the sequel, it is assumed that the cardinality K of \mathcal{X} is known (for ease of notation, \mathcal{X} is set to be $\{1, \dots, K\}$) and that \mathcal{Y} is a subset of \mathbb{R}^D for a positive integer D . Denote by $\mathcal{P}(\mathcal{X})$ the space of probability measures on \mathcal{X} and write \mathcal{L}^D the Lebesgue measure on \mathcal{Y} . For all $n \geq 1$ and all $x \in \mathcal{X}$, the density of the conditional distribution of Y_n given $X_n = x$ with respect to \mathcal{L}^D is written f_x^* . Consider the following assumptions on the hidden chain.

- [H1] a) The transition matrix \mathbf{Q}_* has full rank.
 b) $\delta^* := \min_{1 \leq i, j \leq K} \mathbf{Q}_*(i, j) > 0$.

[H2] The initial distribution $\pi^* := (\pi_1^*, \dots, \pi_K^*)$ is the stationary distribution.

Remark 2.1. Note that under **[H1]-b)** and **[H2]**, for all $k \in \mathcal{X}$, $\pi_k^* \geq \delta^* > 0$.

Remark 2.2. Assumptions **[H1]-a)** and **[H2]** appear in spectral methods, see for instance [3, 18], and in identifiability issues, see for instance [1, 2, 16].

For all $y \in \mathcal{Y}$, define $c_*(y)$ by

$$c_*(y) := \min_{x \in \mathcal{X}} \sum_{x' \in \mathcal{X}} \mathbf{Q}_*(x, x') f_{x'}^*(y). \quad (1)$$

For all $y_{1:n} \in \mathcal{Y}^n$, the filtering distributions $\phi_k^*(\cdot, y_{1:k})$ and marginal smoothing distributions $\phi_{k|n}^*(\cdot, y_{1:n})$ may be computed explicitly for all $1 \leq k \leq n$ using the forward-backward algorithm of [5]. In the forward pass, the filtering distributions ϕ_k^* are updated recursively using, for all $x \in \mathcal{X}$,

$$\phi_1^*(x, y_1) := \frac{\pi^*(x) f_x^*(y_1)}{\sum_{x' \in \mathcal{X}} \pi^*(x') f_{x'}^*(y_1)} \quad \text{and} \quad \phi_k^*(x, y_{1:k}) := \frac{\sum_{x' \in \mathcal{X}} \mathbf{Q}_*(x', x) f_{x'}^*(y_k) \phi_{k-1}^*(x', y_{1:k-1})}{\sum_{x', x'' \in \mathcal{X}} \mathbf{Q}_*(x', x'') f_{x''}^*(y_k) \phi_{k-1}^*(x', y_{1:k-1})}. \quad (2)$$

Note that for all $1 \leq k \leq n$, $\phi_k^*(x, Y_{1:k}) = \mathbb{P}(X_k = x | Y_{1:k})$. In the backward pass, the marginal smoothing distributions may be updated recursively using, for all $x \in \mathcal{X}$,

$$\phi_{n|n}^*(x, y_{1:n}) := \phi_n^*(x, y_{1:n}) \quad \text{and} \quad \phi_{k|n}^*(x, y_{1:n}) := \sum_{x' \in \mathcal{X}} B_{\phi_k^*(\cdot, y_{1:k})}^*(x', x) \phi_{k+1|n}^*(x', y_{1:n}), \quad (3)$$

where, for all $u, v \in \mathcal{X}$ and all $1 \leq k \leq n$,

$$B_{\phi_k^*(\cdot, y_{1:k})}^*(u, v) := \frac{\mathbf{Q}_*(v, u) \phi_k^*(v, y_{1:k})}{\sum_{z \in \mathcal{X}} \mathbf{Q}_*(z, u) \phi_k^*(z, y_{1:k})}.$$

Note that for all $1 \leq k \leq n$, $\phi_{k|n}^*(x, Y_{1:n}) = \mathbb{P}(X_k = x | Y_{1:n})$.

2.2 Preliminary results

In this paper, the parameters π^* , \mathbf{Q}_* and f^* are unknown. Then, the recursive equations (2) and (3) may be applied replacing π^* , \mathbf{Q}_* and f^* by some estimators $\hat{\pi}$, $\hat{\mathbf{Q}}$ and \hat{f} to obtain approximations of the filtering and smoothing distributions. Using forgetting properties of the hidden chain, we are able to obtain an upper bound of the filtering errors and of the marginal smoothing errors by terms involving only the estimation errors of π^* , \mathbf{Q}_* and f^* . These upper bounds are given in propositions 2.1 and 2.2. Their proofs are postponed to Appendix A and B. Note that the upper bounds are given for any possible values $y_{1:k}$, $k \geq 1$ and may be applied to the set of observations for which filtering and smoothing distributions are estimated, whatever the set of observations used to estimate π^* , \mathbf{Q}_* and f^* . Let $\|\cdot\|_{\text{tv}}$ be the total variation norm, $\|\cdot\|_2$ the euclidian norm and $\|\cdot\|_F$ the Frobenius norm. For all $1 \leq k \leq n$, denote by $\hat{\phi}_k$ and $\hat{\phi}_{k|n}$ the approximations of ϕ_k^* and $\phi_{k|n}^*$ obtained by replacing π^* , \mathbf{Q}_* and f^* by the estimators $\hat{\pi}$, $\hat{\mathbf{Q}}$ and \hat{f} in (2) and (3).

Proposition 2.1. Assume **[H1]-b)** and **[H2]** hold. Then, for all $k \geq 1$ and all $y_{1:k} \in \mathcal{Y}^k$,

$$\begin{aligned} \|\phi_k^*(\cdot, y_{1:k}) - \hat{\phi}_k(\cdot, y_{1:k})\|_{\text{tv}} &\leq C_* \left(\rho_*^{k-1} \|\pi^* - \hat{\pi}\|_2 / \delta^* + \|\mathbf{Q}_* - \hat{\mathbf{Q}}\|_F / (\delta^* (1 - \rho_*)) \right. \\ &\quad \left. + \sum_{\ell=1}^k \rho_*^{k-\ell} c_*^{-1}(y_\ell) \max_{x \in \mathcal{X}} |f_x^*(y_\ell) - \hat{f}_x(y_\ell)| \right), \end{aligned}$$

where $\rho_* := 1 - \delta^* / (1 - \delta^*)$ and $C_* := 4(1 - \delta^*) / \delta^*$.

Proposition 2.2. Assume **[H1]-b)** and **[H2]** hold. Then, for all $1 \leq k \leq n$ and all $y_{1:n} \in \mathcal{Y}^n$,

$$\begin{aligned} \|\phi_{k|n}^*(\cdot, y_{1:n}) - \hat{\phi}_{k|n}(\cdot, y_{1:n})\|_{\text{tv}} &\leq C_* \left(\rho_*^{k-1} \|\pi^* - \hat{\pi}\|_2 / \delta^* + [1/(1 - \rho_*) + 1/(1 - \hat{\rho})] \|\mathbf{Q}_* - \hat{\mathbf{Q}}\|_F / \delta^* \right. \\ &\quad \left. + \sum_{\ell=1}^n (\hat{\rho} \vee \rho_*)^{|\ell-k|} c_*^{-1}(y_\ell) \max_{x \in \mathcal{X}} |f_x^*(y_\ell) - \hat{f}_x(y_\ell)| \right), \end{aligned}$$

where $\hat{\delta} := \min_{x, x'} \hat{\mathbf{Q}}(x, x')$ and $\hat{\rho} := 1 - \hat{\delta} / (1 - \hat{\delta})$.

2.3 Uniform consistency of the posterior distributions

Propositions 2.1 and 2.2 are preliminary results that can be used to understand how the estimation errors made on the parameters of the HMM propagate upon the filtering and smoothing distributions. We assume that we are given a set of $p + n$ observations from the hidden Markov model driven by π^* , \mathbf{Q}_* and f^* . The first p observations are used to produce the estimators $\hat{\pi}$, $\hat{\mathbf{Q}}$ and \hat{f} while filtering and smoothing are performed with the last n observations. In other words the estimators $\hat{\pi}$, $\hat{\mathbf{Q}}$ and \hat{f} are measurable functions of $Y_{1:p}$ and the objective is to estimate $\phi_k^*(\cdot, Y_{p+1:p+k})$ and $\phi_{k|n}^*(\cdot, Y_{p+k:p+n})$.

2.3.1 Parametric models

In the parametric case, the hidden Markov model depends on a parameter θ_* which lies in a subset of \mathbb{R}^q for a given $q \geq 1$. In this situation, θ_* may be estimated by $\hat{\theta} \in \mathbb{R}^q$ and we may write $\hat{\pi} := \pi^{\hat{\theta}}$, $\hat{\mathbf{Q}} := Q_{\hat{\theta}}$ and $\hat{f} := f^{\hat{\theta}}$.

Theorem 2.3. Assume [H1] and [H2] hold. Assume also that for all $x, x' \in \mathcal{X}$, $\theta \mapsto Q_{\theta}(x, x')$ is continuously differentiable with a bounded derivative in the neighborhood of θ_* and that for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$, $\theta \mapsto f_x^{\theta}(y)$ is continuously differentiable in the neighborhood of θ_* and such that the norm of its gradient is upper bounded in this neighborhood by a function h_x such that $\int h_x(y) d\mathcal{L}^D(y) < +\infty$. Let $\hat{\theta}$ be a consistent estimator of θ_* . Then for any $1 \leq k \leq n$,

$$\|\phi_k^*(\cdot, Y_{p+1:p+k}) - \hat{\phi}_k(\cdot, Y_{p+1:p+k})\|_{\text{tv}} = O_{\mathbb{P}}(\|\hat{\theta} - \theta_*\|_2)$$

and

$$\|\phi_{k|n}^*(\cdot, Y_{p+1:p+k}) - \hat{\phi}_{k|n}(\cdot, Y_{p+1:p+k})\|_{\text{tv}} = O_{\mathbb{P}}(\|\hat{\theta} - \theta_*\|_2).$$

The smoothness assumption in Theorem 2.3 is usual to study the asymptotic distribution of the maximum likelihood estimator in parametric HMMs. By Theorem 2.3, tight bounds on the uniform convergence rate of $\|\phi_k^*(\cdot, Y_{p+1:p+k}) - \hat{\phi}_k(\cdot, Y_{p+1:p+k})\|_{\text{tv}}$ and of $\|\phi_{k|n}^*(\cdot, Y_{p+1:p+k}) - \hat{\phi}_{k|n}(\cdot, Y_{p+1:p+k})\|_{\text{tv}}$ may be derived by controlling the estimation error $\|\hat{\theta} - \theta_*\|$. There exist several results on this error term depending on the algorithm used to obtain $\hat{\theta}$. For instance, [27] provides explicit upper bounds for this error term in the case where $\hat{\theta}$ is a recursive maximum likelihood estimator of θ_* , under additional assumptions on the model.

Proof. First, under [H1] and [H2], the assumption on $\theta \mapsto Q_{\theta}(x, x')$ implies that $\theta \mapsto \pi_x^{\theta}$ is continuously differentiable with a bounded derivative in the neighborhood of θ_* . Notice also that $\sup_{k \geq 1} \rho_*^{k-1} \leq 1$ and $\sup_{k \geq 1} \hat{\rho}^{k-1} \leq 1$. Then using Taylor expansion we easily get that the first two terms of the upper bound in Propositions 2.1 and 2.2 are $O_{\mathbb{P}}(\|\hat{\theta} - \theta_*\|_2)$. There just remains to control the last term for each of the upper bound in Propositions 2.1 and 2.2. Using a Taylor expansion, Cauchy-Schwarz inequality, and Proposition 2.1, we get that for any $1 \leq k \leq n$,

$$\|\phi_k^*(\cdot, Y_{p+1:p+k}) - \hat{\phi}_k(\cdot, Y_{p+1:p+k})\|_{\text{tv}} \leq O_{\mathbb{P}}(\|\hat{\theta} - \theta_*\|_2) + \|\hat{\theta} - \theta_*\|_2 \sum_{\ell=1}^k \rho_*^{k-\ell} c_*^{-1}(Y_{p+\ell}) \sum_{x \in \mathcal{X}} h_x(Y_{p+\ell}).$$

As the $(Y_j)_{j \geq 1}$ are stationary with distribution having density $\sum_{x \in \mathcal{X}} \pi_x^* f_x^*(y) \leq c_*(y)/\delta^*$, the random variable $\sum_{\ell=1}^k \rho_*^{k-\ell} c_*^{-1}(Y_{p+\ell}) \sum_{x \in \mathcal{X}} h_x(Y_{p+\ell})$ is nonnegative and has expectation upper bounded by

$$\frac{1}{\delta^*} \sum_{\ell=1}^k \rho_*^{k-\ell} \sum_{x \in \mathcal{X}} \int h_x(y) d\mathcal{L}^D(y) \leq \frac{1 - \delta^*}{(\delta^*)^2} \sum_{x \in \mathcal{X}} \int h_x(y) d\mathcal{L}^D(y) < +\infty.$$

Thus $\sum_{\ell=1}^k \rho_*^{k-\ell} c_*^{-1}(Y_{p+\ell}) \sum_{x \in \mathcal{X}} h_x(Y_{p+\ell}) = O_{\mathbb{P}}(1)$ so that we get the first point of Theorem 2.3. The result for the smoothing distributions follows the same lines after noticing that, for some $\epsilon > 0$ such that $\rho_* + \epsilon < 1$, the event $\{\hat{\rho} \geq \rho_* + \epsilon\}$ has probability tending to 0 as p tends to infinity when $\hat{\theta}$ is a consistent estimator of θ_* . \square

2.3.2 Nonparametric models

We first state a general theorem that gives a control of the uniform consistency of the posterior distributions in terms of the risk of the nonparametric estimators. The theorem also holds in the parametric context, however, the parametric literature usually studies the distributional properties of the estimators, while the nonparametric one studies mostly the risk. As usual in the hidden Markov model literature, the model parameters are identifiable up to permutations of the hidden states labels. Therefore, without loss of generality, the following results are stated indicating the prospective permutation of the states. Let \mathcal{S}_K be the set of permutations of $\{1, \dots, K\}$. If τ is a permutation, let \mathbb{P}_τ be the permutation matrix associated with τ .

Theorem 2.4. *Assume [H1]-b) and [H2] hold. Then for all $n \geq 1$, for any permutation $\tau_p \in \mathcal{S}_K$,*

$$\begin{aligned} \sup_{1 \leq k \leq n} \mathbb{E} \left[\left\| \phi_k^*(\cdot, Y_{p+1:p+k}) - \hat{\phi}_k^{\tau_p}(\cdot, Y_{p+1:p+k}) \right\|_{\text{tv}} \right] \\ \leq \frac{C_\star}{(\delta_\star)^2} \left\{ \mathbb{E}[\|\pi^\star - \mathbb{P}_{\tau_p} \hat{\pi}_p\|_2] + \mathbb{E}[\|\mathbf{Q}^\star - \mathbb{P}_{\tau_p} \hat{\mathbf{Q}}_p \mathbb{P}_{\tau_p}^\top\|_F] + \sum_{x \in \mathcal{X}} \mathbb{E}[\|f_x^\star - \hat{f}_{\tau_p(x)}\|_1] \right\} \end{aligned}$$

and

$$\begin{aligned} \sup_{1 \leq k \leq n} \mathbb{E} \left[\left\| \phi_{k|n}^*(\cdot, Y_{p+1:p+n}) - \hat{\phi}_{k|n}^{\tau_p}(\cdot, Y_{p+1:p+n}) \right\|_{\text{tv}} \right] \\ \leq \frac{C_\star}{(\delta_\star)^2} \left\{ \mathbb{E}[\|\pi^\star - \mathbb{P}_{\tau_p} \hat{\pi}_p\|_2] + \mathbb{E}[\|\mathbf{Q}^\star - \mathbb{P}_{\tau_p} \hat{\mathbf{Q}}_p \mathbb{P}_{\tau_p}^\top\|_F / \hat{\delta}] + \sum_{x \in \mathcal{X}} \mathbb{E}[\|f_x^\star - \hat{f}_{\tau_p(x)}\|_1 / \hat{\delta}] \right\}. \end{aligned}$$

Here, $\hat{\phi}_k^{\tau_p}$ and $\hat{\phi}_{k|n}^{\tau_p}$ are the estimation of ϕ_k^\star and $\phi_{k|n}^\star$ based on $\mathbb{P}_{\tau_p} \hat{\mathbf{Q}}_p \mathbb{P}_{\tau_p}^\top$, $\mathbb{P}_{\tau_p} \hat{\pi}_p$ and $\hat{f}_{\tau_p(x)}$, for all $x \in \mathcal{X}$.

Proof. For any $x \in \mathcal{X}$ and any $1 \leq \ell \leq n$,

$$\mathbb{E} \left[c_\star^{-1}(Y_{p+\ell}) \left| f_x^\star(Y_{p+\ell}) - \hat{f}_{\tau_p(x)}(Y_{p+\ell}) \right| \right] = \mathbb{E} \left[\mathbb{E} \left[c_\star^{-1}(Y_{p+\ell}) \left| f_x^\star(Y_{p+\ell}) - \hat{f}_{\tau_p(x)}(Y_{p+\ell}) \right| \middle| Y_{1:p+\ell-1} \right] \right],$$

with

$$\mathbb{E} \left[c_\star^{-1}(Y_{p+\ell}) \left| f_x^\star(Y_{p+\ell}) - \hat{f}_{\tau_p(x)}(Y_{p+\ell}) \right| \middle| Y_{1:p+\ell-1} \right] = \int \left| f_x^\star(z) - \hat{f}_{\tau_p(x)}(z) \right| c_\star^{-1}(z) g_\ell(z) dz,$$

where $g_\ell(z) := \sum_{x_{\ell-1}, x_\ell \in \mathcal{X}} \phi_{\ell-1}^\star(x_{\ell-1}, Y_{p+1:p+\ell-1}) \mathbf{Q}_\star(x_{\ell-1}, x_\ell) f_{x_\ell}^\star(z)$. By [H1]-b) and (1), $c_\star^{-1}(z) g_\ell(z) \leq (1 - \delta_\star) / \delta_\star$ and

$$\mathbb{E} \left[c_\star^{-1}(Y_{p+\ell}) \left| f_x^\star(Y_{p+\ell}) - \hat{f}_{\tau_p(x)}(Y_{p+\ell}) \right| \middle| Y_{1:p+\ell-1} \right] \leq (1 - \delta_\star) \|f_x^\star - \hat{f}_{\tau_p(x)}\|_1 / \delta_\star.$$

Therefore, the result for the filtering distributions comes from taking the supremum and then the expectation in the upper bound of Proposition 2.1. The proof for the smoothing distributions follows the same steps. \square

What comes out in Theorem 2.4 is a control driven by the L^1 -risk of the emission densities. In Section 3, we propose a spectral method to obtain, in the nonparametric context, estimators of the transition matrix, the stationary distribution and the emission densities. The general idea is that of projection methods, so that at the end we obtain a control on the L^2 -risk of the emission densities. This control can be easily transferred whenever \mathcal{Y} is a compact subset of \mathbb{R}^D , since in such a case, for some $C(\mathcal{Y}) > 0$ we have, for any square integrable functions h_1 and h_2 ,

$$\|h_1 - h_2\|_1 \leq C(\mathcal{Y}) \|h_1 - h_2\|_2. \quad (4)$$

We end this section by setting the result that follows when using the spectral estimators. Let $(M_r)_{r \geq 1}$ be an increasing sequence of integers, and let $(\mathfrak{P}_{M_r})_{r \geq 1}$ be a sequence of nested subspaces such that their union is dense in $L^2(\mathcal{Y}, \mathcal{L}^D)$. Let $\Phi_{M_r} := \{\varphi_1, \dots, \varphi_{M_r}\}$ be an orthonormal basis of \mathfrak{P}_{M_r} . Note that for all $f \in L^2(\mathcal{Y}, \mathcal{L}^D)$,

$$\lim_{p \rightarrow \infty} \sum_{m=1}^{M_p} \langle f, \varphi_m \rangle \varphi_m = f, \quad (5)$$

in $L^2(\mathcal{Y}, \mathcal{L}^D)$. Note also that changing M_r may change all functions φ_r , $1 \leq m \leq M_r$ in the basis Φ_{M_r} , which will not be indicated in the notation for better clarity. We shall also drop the index r and write M instead of M_r . The spectral estimators of the emission densities will be projection estimators. Let us denote $f_{M,1}^*, \dots, f_{M,K}^*$ the projections of the emission densities on the space \mathfrak{P}_M , that is, for $x \in \mathcal{X}$,

$$f_{M,x}^* = \sum_{m=1}^M \langle f_x^*, \varphi_m \rangle \varphi_m.$$

We need a further assumption, which, together with **[H1]-b)** and **[H2]**, has been proved sufficient to get identifiability in nonparametric HMMs, see [16].

[H3] The family of emission densities $\mathfrak{F}^* := \{f_1^*, \dots, f_K^*\}$ is linearly independent.

Finally, the following quantity is needed in the control of the L^2 -risk of the spectral estimators. For any M , define

$$\eta_3^2(\Phi_M) := \sup_{y, y' \in \mathcal{Y}^3} \sum_{a,b,c=1}^M (\varphi_a(y_1)\varphi_b(y_2)\varphi_c(y_3) - \varphi_a(y'_1)\varphi_b(y'_2)\varphi_c(y'_3))^2. \quad (6)$$

Applying Theorem 2.4 and (4) we get the following corollary whose proof is omitted: the first point is an application of Corollary 3.2, the second point is obtained following the same lines as the proof of Corollary 3.2.

Corollary 2.5. *Assume **[H1]-[H3]** hold. Assume also that for all $x \in \mathcal{X}$, $f_x^* \in L^2(\mathcal{Y}, \mathcal{L}^D)$. Let M_p be a sequence of integers tending to infinity such that $\eta_3(\Phi_{M_p}) = o(\sqrt{p/\log p})$. For each p , define \hat{f} , $\hat{\mathbf{Q}}$ and $\hat{\pi}$ as the estimators obtained by the spectral algorithm given in Section 3 with this choice of M_p . Then, there exists a sequence of permutations $\tau_p \in \mathcal{S}_K$ such that*

$$\mathbb{E} \left[\sup_{k \geq 1} \|\phi_k^*(\cdot, Y_{p+1:p+k}) - \hat{\phi}_{k|\tau_p}^{\tau_p}(\cdot, Y_{p+1:p+k})\|_{\text{tv}} \right] = O(\eta_3(\Phi_{M_p})\sqrt{\log p/p} + \sum_{x \in \mathcal{X}} \|f_x^* - f_{M_p,x}^*\|_2)$$

and

$$\mathbb{E} \left[\sup_{1 \leq k \leq n} \|\phi_{k|n}^*(\cdot, Y_{p+1:p+n}) - \hat{\phi}_{k|n}^{\tau_p}(\cdot, Y_{p+1:p+n})\|_{\text{tv}} \right] = O(\eta_3(\Phi_{M_p})\sqrt{\log p/p} + \sum_{x \in \mathcal{X}} \|f_x^* - f_{M_p,x}^*\|_2).$$

One may consider the following standard examples.

- **(Spline)** The space of piecewise polynomials of degree bounded by d_r based on the regular partition with p_r^D regular pieces on \mathcal{Y} . It holds that $M_r = (d_r + 1)^D p_r^D$.
- **(Trig.)** The space of real trigonometric polynomials on \mathcal{Y} with degree less than r . It holds that $M_r = (2r + 1)^D$.
- **(Wav.)** A wavelet basis Φ_{M_r} of scale r on \mathcal{Y} , see [22]. It holds that $M_r = 2^{(r+1)D}$.

In those examples, there exists a constant $C_\eta > 0$ such that $\eta_3(M) \leq C_\eta M^{k/2}$, so that the rate of uniform convergence for the posterior probabilities is $O(M_p^{3/2} \sqrt{\log p/p} + \sum_{x \in \mathcal{X}} \|f_x^* - f_{M_p,x}^*\|_2)$.

3 Nonparametric spectral estimation of HMMs

3.1 Description of the spectral method

This section describes a tractable approach to get nonparametric estimators of the emission densities and of the transition matrix. Our procedure relies on the estimation of the projections of the emission laws onto nested subspaces of increasing complexity. This allows to illustrate the uniform consistency result provided in the previous section.

Recall that $(\mathfrak{P}_{M_r})_{r \geq 1}$ is a sequence of nested subspaces of $L^2(\mathcal{Y}, \mathcal{L}^D)$ associated with their orthonormal basis $(\Phi_{M_r})_{r \geq 1}$. Since projections are linear functionals of the distributions, it is possible to use spectral methods to estimate the projections of the emission distributions on the basis Φ_M for each M . To this end, our approach is based on the work described in [3]. In particular, we follow their strategy to get an estimation of the

emission densities. However, the dependency in the dimension is of crucial importance in the nonparametric framework and it has not been addressed in [3]. Hence, we present in Theorem C.3 a new quantitative version of the work [3] that accounts for the dimension M . Moreover, the authors of [3] invoke a way of estimating the transition matrix \mathbf{Q}_* but they do not give any theoretical guarantees regarding this estimator. In this paper, we introduce a slightly different estimator that is based on a surrogate $\hat{\pi}$ (see Step 8 of Algorithm 1) of the stationary distribution. Our estimator (see Step 9 of Algorithm 1) is then build from the "observable" operator (rather than its left singular vectors as done in [3]). Eventually, Theorem C.2 gives the theoretical guarantees of our estimator of the transition matrix and its stationary distribution.

The computation of those estimators is particularly simple: it is based on one singular value decomposition, matrix inversions and one diagonalization. It is proved in Theorem C.2 and C.3 that, with overwhelming probability, all the matrix inversions and the diagonalization can be done rightfully.

For all $(p \times q)$ matrix A with $p \geq q$, denote by $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_q(A) \geq 0$ its singular values and $\|\cdot\|$ its operator norm. When A is invertible, let $\kappa(A) := \sigma_1(A)/\sigma_q(A)$ be its condition number. A^\top is the transpose matrix of A , $A(\ell, \ell')$ its (ℓ, ℓ') th entry, $A(\cdot, \ell)$ its ℓ th column and $A(k, \cdot)$ its k th line. When A is a $(p \times p)$ diagonalizable matrix, its eigenvalues are written $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_p(A)$. For any $1 \leq q \leq +\infty$, $\|\cdot\|_q$ is the usual L^q norm for vectors. For any row or column vector v , denote by $\text{Diag}[v]$ the diagonal matrix with diagonal entries v_i . The following vectors, matrices and tensors are used throughout the paper:

- $\mathbf{L}_M \in \mathbb{R}^M$ is the projection of the distribution of one observation on the basis Φ_M : for all $a \in \{1, \dots, M\}$, $\mathbf{L}_M(a) := \mathbb{E}[\varphi_a(Y_1)]$;
- $\mathbf{N}_M \in \mathbb{R}^{M \times M}$ is the joint distribution of two consecutive observations: for all $(a, b) \in \{1, \dots, M\}^2$, $\mathbf{N}_M(a, b) := \mathbb{E}[\varphi_a(Y_1)\varphi_b(Y_2)]$;
- $\mathbf{M}_M \in \mathbb{R}^{M \times M \times M}$ is the joint distribution of three consecutive observations: for all $(a, b, c) \in \{1, \dots, M\}^3$, $\mathbf{M}_M(a, b, c) := \mathbb{E}[\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)]$;
- $\mathbf{O}_M \in \mathbb{R}^{M \times K}$ is the conditional distribution of one observation on the basis Φ_M : for all $(m, x) \in \{1, \dots, M\} \times \mathcal{X}$, $\mathbf{O}_M(m, x) := \mathbb{E}[\varphi_m(Y_1)|X_1 = x] = \langle f_x^*, \varphi_m \rangle$;
- For all $x \in \mathcal{X}$, $f_{M,x}^*$ is the projection of the emission laws on the subspace \mathfrak{P}_M : , $f_{M,x}^* := \sum_{m=1}^M \mathbf{O}_M(m, x)\varphi_m$. Write $\mathbf{f}_M^* := (f_{M,1}^*, \dots, f_{M,K}^*)$;
- $\mathbf{P}_M \in \mathbb{R}^{M \times M}$ is the joint distribution of (Y_1, Y_3) : for all $(a, c) \in \{1, \dots, M\}^2$, $\mathbf{P}_M(a, c) := \mathbb{E}[\varphi_a(Y_1)\varphi_c(Y_3)]$.

3.2 Variance of the spectral estimators

This section displays the results which allow to derive the asymptotic properties of the spectral estimators. The aim of Theorem 3.1 is to provide an upper bound for the variance term with an explicit dependency with respect to both p and M . The way it depends in M is described by the quantity η_3 defined in (6). Recall that, in the examples (**Spline**), (**Trig.**) and (**Wav.**), we have $\eta_3(\Phi_M) \leq C_\eta M^{3/2}$ with $C_\eta > 0$ a constant. In this section, assumption **[H1]** may be replaced by the following weaker assumption **[H1']**.

- [H1']** a) The transition matrix \mathbf{Q}_* has full rank.
b) $(X_n)_{n \geq 1}$ is irreducible and aperiodic.

Note that under **[H1']** and **[H2]**, there exists $\pi_{\min}^* > 0$ such that, for all $x \in \mathcal{X}$,

$$\pi_x^* \geq \pi_{\min}^*. \quad (7)$$

Theorem 3.1 (Spectral estimators). *Assume that **[H1']** and **[H2]-[H3]** hold. Assume also that for all $x \in \mathcal{X}$, $f_x^* \in L^2(\mathcal{Y}, \mathcal{L}^D)$. Then, there exist positive constant $u(\mathbf{Q}^*)$, $\mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)$ and $\mathbf{N}(\mathbf{Q}^*, \mathfrak{F}^*)$ such that for any $u \geq u(\mathbf{Q}^*)$, any $\delta \in (0, 1)$, any $M \geq M_{\mathfrak{F}^*}$, there exists a permutation $\tau_M \in \mathcal{S}_K$ such that the spectral method estimators $\hat{f}_{M,x}$, $\hat{\pi}$ and $\hat{\mathbf{Q}}$ (see Algorithm 1) satisfy, for any $p \geq \mathbf{N}(\mathbf{Q}^*, \mathfrak{F}^*)\eta_3(\Phi_M)^2 u(-\log \delta)/\delta^2$, with probability greater than $1 - 2\delta - 4e^{-u}$,*

$$\max_{x \in \mathcal{X}} \|f_{M,x}^* - \hat{f}_{M,\tau_M(x)}\|_2 \leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \frac{\sqrt{-\log \delta}}{\delta} \frac{\eta_3(\Phi_M)}{\sqrt{p}} \sqrt{u},$$

Algorithm 1: Nonparametric spectral estimation of the transition matrix and the emission laws

Data: An observed chain (Y_1, \dots, Y_{p+2}) and a number of hidden states K .

Result: Spectral estimators $\hat{\pi}$, $\hat{\mathbf{Q}}$ and $(\hat{f}_{M,x})_{x \in \mathcal{X}}$.

[Step 1] For all a, b, c in $\{1, \dots, M\}$, consider the following empirical estimators:

$$\begin{aligned}\hat{\mathbf{L}}_M(a) &:= \sum_{s=1}^p \varphi_a(Y_s)/p, \quad \hat{\mathbf{M}}_M(a, b, c) := \sum_{s=1}^p \varphi_a(Y_s) \varphi_b(Y_{s+1}) \varphi_c(Y_{s+2})/p, \\ \hat{\mathbf{N}}_M(a, b) &:= \sum_{s=1}^p \varphi_a(Y_s) \varphi_b(Y_{s+1})/p \text{ and } \hat{\mathbf{P}}_M(a, c) := \sum_{s=1}^p \varphi_a(Y_s) \varphi_c(Y_{s+2})/p.\end{aligned}$$

[Step 2] Let $\hat{\mathbf{U}}$ be the $M \times K$ matrix of orthonormal right singular vectors of $\hat{\mathbf{P}}_M$ corresponding to its top K singular values.

[Step 3] For all $b \in \{1, \dots, M\}$, set $\hat{\mathbf{B}}(b) := (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{M}}_M(\cdot, b, \cdot) \hat{\mathbf{U}}$.

[Step 4] Set Θ a $(K \times K)$ unitary matrix uniformly drawn and, $\forall x \in \mathcal{X}$, $\hat{\mathbf{C}}(x) := \sum_{b=1}^M (\hat{\mathbf{U}} \Theta)(b, x) \hat{\mathbf{B}}(b)$.

[Step 5] Compute $\hat{\mathbf{R}}$ a $(K \times K)$ unit Euclidean norm columns matrix that diagonalizes the matrix $\hat{\mathbf{C}}(1)$:

$$\hat{\mathbf{R}}^{-1} \hat{\mathbf{C}}(1) \hat{\mathbf{R}} = \mathfrak{D}\text{diag}[(\hat{\Lambda}(1, 1), \dots, \hat{\Lambda}(1, K))].$$

[Step 6] For all $x, x' \in \mathcal{X}$, set $\hat{\Lambda}(x, x') := (\hat{\mathbf{R}}^{-1} \hat{\mathbf{C}}(x) \hat{\mathbf{R}})(x', x')$ and $\hat{\mathbf{O}}_M := \hat{\mathbf{U}} \Theta \hat{\Lambda}$.

[Step 7] Consider the estimator $(\hat{f}_{M,x})_{x \in \mathcal{X}}$ defined by, for all $x \in \mathcal{X}$, $\hat{f}_{M,x} := \sum_{m=1}^M \hat{\mathbf{O}}_M(m, x) \varphi_m$.

[Step 8] Set $\tilde{\pi} := (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M$.

[Step 9] Consider the transition matrix estimator $\hat{\mathbf{Q}} := \Pi_{\text{TM}} \left((\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M \mathfrak{D}\text{diag}[\tilde{\pi}])^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{N}}_M \hat{\mathbf{U}} (\hat{\mathbf{O}}_M^\top \hat{\mathbf{U}})^{-1} \right)$ where Π_{TM} denotes the projection (with respect to the scalar product given by the Frobenius norm) onto the convex set of transition matrices, and define $\hat{\pi}$ as the stationary distribution of $\hat{\mathbf{Q}}$.

$$\begin{aligned}\|\pi^* - \mathbb{P}_{\tau_M} \hat{\pi}\|_2 &\leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \frac{\sqrt{-\log \delta} \eta_3(\Phi_M)}{\delta \sqrt{p}} \sqrt{u}, \\ \|\mathbf{Q}^* - \mathbb{P}_{\tau_M} \hat{\mathbf{Q}} \mathbb{P}_{\tau_M}^\top\| &\leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \frac{\sqrt{-\log \delta} \eta_3(\Phi_M)}{\delta \sqrt{p}} \sqrt{u}.\end{aligned}$$

Corollary 3.2. Assume that **[H1']** and **[H2]-[H3]** hold. Assume also that for all $x \in \mathcal{X}$, $f_x^* \in \mathcal{L}^2(\mathcal{Y}, \mathcal{L}^D)$. Let M_p be a sequence of integers tending to infinity and such that $\eta_3(\Phi_{M_p}) = o(\sqrt{p/\log p})$. For each p , define \hat{f} , $\hat{\mathbf{Q}}$ and $\hat{\pi}$ as the estimators obtained by the spectral algorithm with this choice of M_p . Then, there exists a sequence of permutations $\tau_p \in \mathcal{S}_K$ such that

$$\mathbb{E} \left[\max_{x \in \mathcal{X}} \|f_{M_p, x}^* - \hat{f}_{\tau_p(x)}\|_2 \right] \vee \mathbb{E} [\|\mathbf{Q}^* - \mathbb{P}_{\tau_p} \hat{\mathbf{Q}} \mathbb{P}_{\tau_p}^\top\|] \vee \mathbb{E} [\|\pi^* - \mathbb{P}_{\tau_p} \hat{\pi}\|_2] = O(\eta_3(\Phi_{M_p}) \sqrt{\log p/p}) = o(1).$$

Here, the expectations are with respect to the observations and to the random unitary matrix drawn at **[Step 4]** of Algorithm 1.

Proof. Apply Theorem 3.1 where, for each p , we define δ_p such that $(-\log \delta_p)/\delta_p^2 := \log p$. δ_p goes to 0 and M_p goes to infinity as p tends to infinity so that for any large enough p , $M_p \geq M_{\mathfrak{F}^*}$. Let τ_p the permutation τ_{M_p} given by Theorem 3.1. Then, for all $p/(\mathbf{N}(\mathbf{Q}^*, \mathfrak{F}^*) \eta_3(\Phi_{M_p})^2 \log p) \geq u \geq u(\mathbf{Q}^*)$, with probability $1 - 4e^{-u} - 2\delta_p$,

$$\max_{x \in \mathcal{X}} \|f_{M, x}^* - \hat{f}_{M, \tau_M(x)}\|_2 \vee \|\pi^* - \mathbb{P}_{\tau_p} \hat{\pi}\|_2 \vee \|\mathbf{Q}^* - \mathbb{P}_{\tau_p} \hat{\mathbf{Q}} \mathbb{P}_{\tau_p}^\top\| \leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \eta_3(\Phi_{M_p}) \sqrt{\log p/p} \sqrt{u}.$$

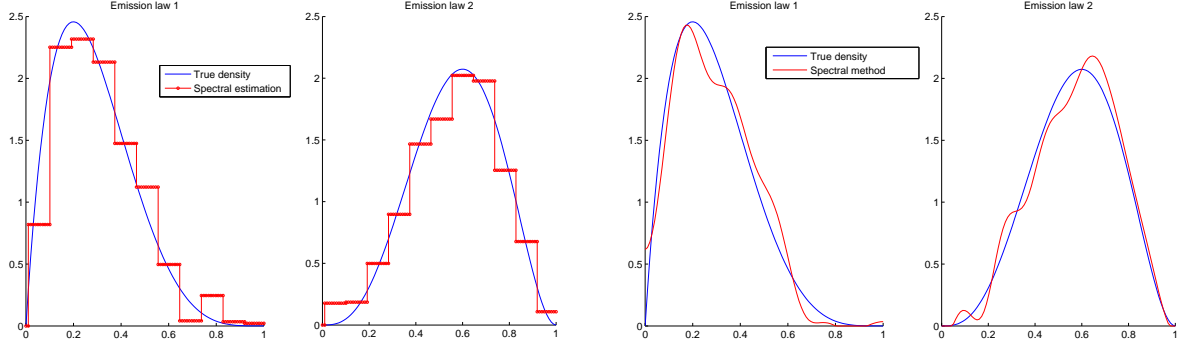


Figure 1: Estimation of emission laws of beta distributions with parameters (2, 5) and (4, 3) using the spectral method. The projection basis is the histogram basis (left panel) or the trigonometric basis (right panel).

It yields

$$\begin{aligned}
& \limsup_{p \rightarrow +\infty} \mathbb{E} \left[\frac{p}{\eta_3(\Phi_{M_p})^2 \log p} \|\mathbf{Q}^* - \mathbb{P}_{\tau_p} \hat{\mathbf{Q}} \mathbb{P}_{\tau_p}^\top\|^2 \right] \\
& \leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)^2 \int_0^{+\infty} \limsup_{p \rightarrow +\infty} \mathbb{P} \left(\frac{\sqrt{p}}{\mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \eta_3(\Phi_{M_p}) \sqrt{\log p}} \|\mathbf{Q}^* - \mathbb{P}_{\tau_p} \hat{\mathbf{Q}} \mathbb{P}_{\tau_p}^\top\| \geq \sqrt{u} \right) du \\
& \leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)^2 u(\mathbf{Q}^*) + \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)^2 \int_{x(\mathbf{Q}^*)}^{+\infty} 4e^{-u} du < +\infty.
\end{aligned}$$

The proof is similar for the other terms. \square

4 Experimental results

We have run several numerical experiments to assess the efficiency of our method. We consider $K = 2$ emission laws of beta distributions with parameters (2, 5) and (4, 3). In all our experiments, the transition matrix \mathbf{Q}_* is given by

$$\mathbf{Q}_* := \begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix}.$$

We observe a sequence of $n = 6 \times 10^4$ variables $(Y_i)_{i=1}^n$. As projection basis, we have considered the histogram basis or the trigonometric basis. The minimax adaptive procedure described in [7] gives an estimation of \mathbf{Q}_* and of the emission laws. Using the slope heuristic [4], we find that the selected size of the model is $\hat{M} = 11$ in the histogram case and $\hat{M} = 13$ in the trigonometric case. Figure 1 presents the adaptive estimation of the emission laws. From these estimates, we compute an estimation of the marginal smoothing probabilities using the forward-backward algorithm. The results are presented in Figure 2.

A Control of the filtering error - Proof of Proposition 2.1

Let $y_{1:n} \in \mathcal{Y}^n$. The aim of this section consists in establishing that the total variation error between $\phi_k^*(\cdot, y_{1:n})$ and its approximations based on $\hat{\mathbf{Q}}$ and \hat{f} is bounded uniformly in time k . Before stating the main result, we introduce a standard decomposition of the filtering error $\phi_k^*(\cdot, y_{1:k}) - \hat{\phi}_k(\cdot, y_{1:k})$. For all $k \geq 1$, let F_{k,y_k}^* be the forward kernel at time k and \hat{F}_{k,y_k} its approximation, defined, for all $\nu \in \mathcal{P}(\mathcal{X})$, as:

$$F_{k,y_k}^* \nu(x) := \frac{\sum_{x' \in \mathcal{X}} \mathbf{Q}_*(x', x) f_x^*(y_k) \nu(x')}{\sum_{x', x'' \in \mathcal{X}} \mathbf{Q}_*(x', x'') f_{x''}^*(y_k) \nu(x')},$$

and

$$\hat{F}_{k,y_k} \nu(x) := \frac{\sum_{x' \in \mathcal{X}} \hat{\mathbf{Q}}(x', x) \hat{f}_x(y_k) \nu(x')}{\sum_{x', x'' \in \mathcal{X}} \hat{\mathbf{Q}}(x', x'') \hat{f}_{x''}(y_k) \nu(x')}.$$

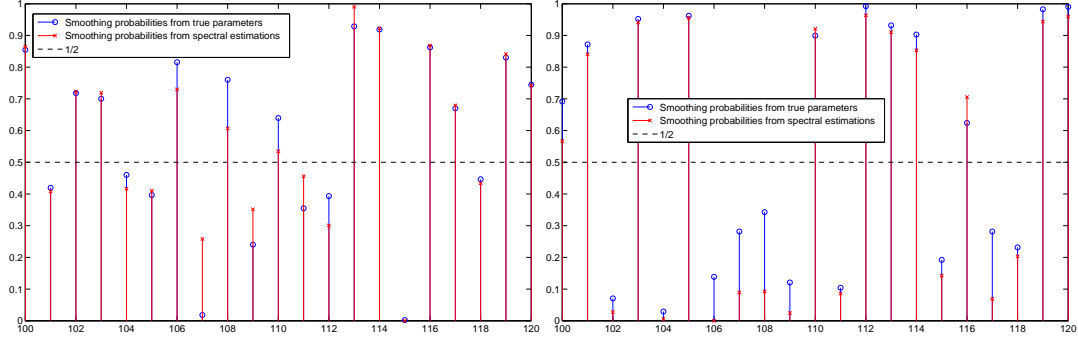


Figure 2: Marginal smoothing probabilities obtained with the forward-backward algorithm combined with the spectral method using projection of the emission laws on the histogram basis (top panel) or the trigonometric basis (bottom panel).

Clearly, for all $y_{1:n} \in \mathcal{Y}^n$ and $2 \leq k \leq n$, $\phi_k^*(\cdot, y_{1:k}) = F_{k,y_k}^* \phi_{k-1}^*(\cdot, y_{1:k-1})$ and $\hat{\phi}_k(\cdot, y_{1:k}) = \hat{F}_{k,y_k} \hat{\phi}_{k-1}(\cdot, y_{1:k-1})$. The filtering error is usually written as a sum of one step errors. For all $k \geq 2$,

$$\begin{aligned} \phi_k^*(\cdot, y_{1:k}) - \hat{\phi}_k(\cdot, y_{1:k}) &= F_{k,y_k}^* \phi_{k-1}^*(\cdot, y_{1:k-1}) - \hat{F}_{k,y_k} \hat{\phi}_{k-1}(\cdot, y_{1:k-1}) \\ &= \sum_{\ell=1}^{k-1} \Delta_{k,\ell}(y_{\ell:k}) + F_{k,y_k}^* \hat{\phi}_{k-1}(\cdot, y_{1:k-1}) - \hat{F}_{k,y_k} \hat{\phi}_{k-1}(\cdot, y_{1:k-1}), \end{aligned} \quad (8)$$

with $F_{1,y_1}^* \hat{\phi}_0 = \phi_1^*(\cdot, y_1)$ and

$$\Delta_{k,\ell}(y_{\ell:k}) := F_{k,y_k}^* \dots F_{\ell+1,y_{\ell+1}}^* F_{\ell,y_\ell}^* \hat{\phi}_{\ell-1}(\cdot, y_{1:\ell-1}) - F_{k,y_k}^* \dots F_{\ell+1,y_{\ell+1}}^* \hat{\phi}_\ell(\cdot, y_\ell).$$

Let $\beta_{\ell|k}^*[y_{\ell+1:k}]$ and $F_{\ell|k}^*[y_{\ell:k}]$ be the backward functions and the forward smoothing transition matrix as defined in [6, Chapter 3],

$$\beta_{\ell|k}^*[y_{\ell+1:k}](x_\ell) := \sum_{x_{\ell+1:k}} \mathbf{Q}_*(x_\ell, x_{\ell+1}) f_{x_{\ell+1}}^*(y_{\ell+1}) \dots \mathbf{Q}_*(x_{k-1}, x_k) f_{x_k}^*(y_k), \quad (9)$$

$$F_{\ell|k}^*[y_{\ell:k}](x_{\ell-1}, x_\ell) := \frac{\beta_{\ell|k}^*[y_{\ell+1:k}](x_\ell) \mathbf{Q}_*(x_{\ell-1}, x_\ell) f_{x_\ell}^*(y_\ell)}{\sum_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x) \mathbf{Q}_*(x_{\ell-1}, x) f_x^*(y_\ell)}. \quad (10)$$

In the sequel, the dependency on the observations may be dropped to simplify notations. By [6, Chapter 4], for any probability distribution ν , $F_k^* \dots F_{\ell+1}^* \nu = \nu_{\ell|k} F_{\ell+1|k}^* \dots F_{k|k}^*$, where $\nu_{\ell|k} \propto \beta_{\ell|k}^* \nu$. Therefore, the filtering error (8) is given by:

$$\phi_k^* - \hat{\phi}_k = \sum_{\ell=1}^{k-1} \left(\mu_{\ell|k}^* F_{\ell+1|k}^* \dots F_{k|k}^* - \hat{\mu}_{\ell|k} F_{\ell+1|k}^* \dots F_{k|k}^* \right) + F_k^* \hat{\phi}_{k-1} - \hat{F}_k \hat{\phi}_{k-1}, \quad (11)$$

where $\mu_{\ell|k}^* \propto \beta_{\ell|k}^* F_\ell^* \hat{\phi}_{\ell-1}$ and $\hat{\mu}_{\ell|k} \propto \beta_{\ell|k} \hat{\phi}_\ell$. By **[H1-b)**, the transition matrix $F_{k|n}^*$ can be lower bounded uniformly in its first component:

$$F_{\ell|k}^*(x, x') \geq \frac{\delta^*}{1 - \delta^*} \frac{\beta_{\ell|k}^*[y_{\ell+1:k}](x') f_{x'}^*(y_\ell)}{\sum_{z \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](z) f_z^*(y_\ell)}.$$

By [6, Chapter 4], this allows to write,

$$\left\| \mu_{\ell|k}^* F_{\ell+1|k}^* \dots F_{k|k}^* - \hat{\mu}_{\ell|k} F_{\ell+1|k}^* \dots F_{k|k}^* \right\|_{\text{tv}} \leq \rho_{\star}^{k-\ell} \left\| \mu_{\ell|k}^* - \hat{\mu}_{\ell|k} \right\|_{\text{tv}}. \quad (12)$$

Eq. (12) is the crucial step to obtain the upper bound for the filtering error stated in Proposition 2.1. By (11) and (12),

$$\left\| \phi_k^* - \hat{\phi}_k \right\|_{\text{tv}} \leq \sum_{\ell=1}^{k-1} \rho_{\star}^{k-\ell} \left\| \mu_{\ell|k}^* - \hat{\mu}_{\ell|k} \right\|_{\text{tv}} + \left\| F_k^* \hat{\phi}_{k-1} - \hat{F}_k \hat{\phi}_{k-1} \right\|_{\text{tv}}.$$

For all $1 \leq \ell \leq k-1$ and all bounded function h on \mathcal{X} , $\left| \mu_{\ell|k}^*(h) - \hat{\mu}_{\ell|k}(h) \right| \leq T_1 + T_2$ where

$$T_1 := \left| \frac{\sum_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x) h(x) \left[F_{\ell}^* \hat{\phi}_{\ell-1}(x) - \hat{F}_{\ell} \hat{\phi}_{\ell-1}(x) \right]}{\sum_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x) F_{\ell}^* \hat{\phi}_{\ell-1}(x)} \right|,$$

$$T_2 := \left| \frac{\sum_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x) h(x) \hat{F}_{\ell} \hat{\phi}_{\ell-1}(x)}{\sum_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x) \hat{F}_{\ell} \hat{\phi}_{\ell-1}(x)} \right| \cdot \left| \frac{\sum_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x) \left[F_{\ell}^* \hat{\phi}_{\ell-1}(x) - \hat{F}_{\ell} \hat{\phi}_{\ell-1}(x) \right]}{\sum_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x) F_{\ell}^* \hat{\phi}_{\ell-1}(x)} \right|.$$

Both T_1 and T_2 are upper bounded by the same term so that

$$T_1 + T_2 \leq 2 \frac{\|h\|_{\infty} \cdot \|\beta_{\ell|k}^*[y_{\ell+1:k}]\|_{\infty}}{\inf_{x \in \mathcal{X}} \beta_{\ell|k}^*[y_{\ell+1:k}](x)} \|F_{\ell}^* \hat{\phi}_{\ell-1} - \hat{F}_{\ell} \hat{\phi}_{\ell-1}\|_{\text{tv}}.$$

By (9), for all $x \in \mathcal{X}$, $\beta_{\ell|k}^*[y_{\ell+1:k}](x) \leq (1-\delta^*) \sum_{x_{k+1:n}} f_{x_{k+1}}^*(y_{k+1}) \dots \mathbf{Q}_{\star}(x_{n-1}, x_n) f_{x_n}^*(y_n)$ and $\beta_{\ell|k}^*[y_{\ell+1:k}](x) \geq \delta^* \sum_{x_{k+1:n}} f_{x_{k+1}}^*(y_{k+1}) \dots \mathbf{Q}_{\star}(x_{n-1}, x_n) f_{x_n}^*(y_n)$, showing that

$$T_1 + T_2 \leq 2 \|h\|_{\infty} \left(\frac{1-\delta^*}{\delta^*} \right) \|F_{\ell}^* \hat{\phi}_{\ell-1} - \hat{F}_{\ell} \hat{\phi}_{\ell-1}\|_{\text{tv}}.$$

Now, for all $2 \leq \ell \leq k$ and all bounded function h on \mathcal{X} , $\left| F_{\ell}^* \hat{\phi}_{\ell-1}(h) - \hat{F}_{\ell} \hat{\phi}_{\ell-1}(h) \right| \leq R_1 + R_2$, where

$$R_1 := \left| \frac{\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \left[\mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell}) - \hat{\mathbf{Q}}(x, x') \hat{f}_{x'}(y_{\ell}) \right] h(x')}{\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell})} \right|,$$

$$R_2 := \left| \frac{\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \hat{\mathbf{Q}}(x, x') \hat{f}_{x'}(y_{\ell}) h(x')}{\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \hat{\mathbf{Q}}(x, x') \hat{f}_{x'}(y_{\ell})} \right|$$

$$\times \left| \frac{\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \left[\mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell}) - \hat{\mathbf{Q}}(x, x') \hat{f}_{x'}(y_{\ell}) \right]}{\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell})} \right|.$$

Then,

$$R_1 \leq \left(\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell}) \right)^{-1} \sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \left| \mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell}) - \hat{\mathbf{Q}}(x, x') \hat{f}_{x'}(y_{\ell}) \right| h(x'),$$

$$\leq \left(\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell}) \right)^{-1} \sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \left| \mathbf{Q}_{\star}(x, x') - \hat{\mathbf{Q}}(x, x') \right| f_{x'}^*(y_{\ell}) h(x')$$

$$+ \left(\sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \mathbf{Q}_{\star}(x, x') f_{x'}^*(y_{\ell}) \right)^{-1} \sum_{x, x' \in \mathcal{X}} \hat{\phi}_{\ell-1}(x) \hat{\mathbf{Q}}(x, x') \left| f_{x'}^*(y_{\ell}) - \hat{f}_{x'}(y_{\ell}) \right| h(x'),$$

$$\leq \|h\|_{\infty} \left[\|\mathbf{Q}_{\star} - \hat{\mathbf{Q}}\|_F / \delta^* + c_{\star}^{-1}(y_{\ell}) \max_{x \in \mathcal{X}} \left| f_x^*(y_{\ell}) - \hat{f}_x(y_{\ell}) \right| \right],$$

where c_{\star} is defined in (1). The same upper bound holds for R_2 . In the case $\ell = 1$,

$$\left\| F_1^* \hat{\phi}_0 - \hat{\phi}_1 \right\|_{\text{tv}} \leq \left\| \phi_1^* - \hat{\phi}_1 \right\|_{\text{tv}} \leq 2 \left[\|\pi^* - \hat{\pi}\|_2 / \delta^* + c_{\star}^{-1}(y_1) \max_{x \in \mathcal{X}} \left| f_x^*(y_1) - \hat{f}_x(y_1) \right| \right].$$

Therefore, the filtering error is upper bounded as follows:

$$\|\phi_k^* - \hat{\phi}_k\|_{\text{tv}} \leq 4 \left(\frac{1-\delta^*}{\delta^*} \right) \sum_{\ell=2}^k \rho_{\star}^{k-\ell} \left[\|\mathbf{Q}_{\star} - \hat{\mathbf{Q}}\|_F / \delta^* + c_{\star}^{-1}(y_{\ell}) \max_{x \in \mathcal{X}} \left| f_x^*(y_{\ell}) - \hat{f}_x(y_{\ell}) \right| \right]$$

$$+ 4 \left(\frac{1-\delta^*}{\delta^*} \right) \rho_{\star}^{k-1} \left[\|\pi^* - \hat{\pi}\|_2 / \delta^* + c_{\star}^{-1}(y_1) \max_{x \in \mathcal{X}} \left| f_x^*(y_1) - \hat{f}_x(y_1) \right| \right].$$

B Control of the marginal smoothing error - Proof of Proposition 2.2

Let $y_{1:n} \in \mathcal{Y}^n$. The aim of this section consists in establishing that the total variation error between $\phi_{k|n}^*(\cdot, y_{1:n})$ and its approximations based on $\hat{\mathbf{Q}}$ and \hat{f} is bounded uniformly in time k . Before stating the main result, we display the decomposition of the smoothing error $\phi_{k|n}^*(\cdot, y_{1:n}) - \hat{\phi}_{k|n}(\cdot, y_{1:n})$ depicted in [10] and used in [14] to obtain nonasymptotic upper bounds for the marginal smoothing error when $\phi_{k|n}^*(\cdot, y_{1:n})$ is approximated using Sequential Monte Carlo methods. In the sequel, the dependency on the observations may be dropped to simplify notations. For any bounded function h on \mathcal{X}^n , $\phi_{1:n|n}^*(h)$ can be written, for any $1 \leq \ell \leq n$

$$\phi_{1:n|n}^*(h) = \frac{\phi_{1:\ell|n}^*(L_{\ell,n}^*(\cdot, h))}{\phi_{1:\ell|n}^*(L_{\ell,n}^*(\cdot, \mathbb{1}))},$$

where $\mathbb{1}$ is the constant function which equals 1 and, for all $x_{1:\ell} \in \mathcal{X}^\ell$,

$$L_{\ell,n}^*(x_{1:\ell}, h) := \sum_{x_{\ell+1:n} \in \mathcal{X}^{n-\ell}} \prod_{u=\ell+1}^n \mathbf{Q}_*(x_{u-1}, x_u) f_{x_u}^*(y_u) h(x_{1:n}). \quad (13)$$

As for the filtering error, the smoothing error can be decomposed as a telescopic sum of one step errors:

$$\begin{aligned} \hat{\phi}_{1:n|n}(h) - \phi_{1:n|n}^*(h) &= \sum_{\ell=2}^n \left(\frac{\hat{\phi}_{1:\ell|n}(L_{\ell,n}^*(\cdot, h))}{\hat{\phi}_{1:\ell|n}(L_{\ell,n}^*(\cdot, \mathbb{1}))} - \frac{\hat{\phi}_{1:\ell-1|n}(L_{\ell-1,n}^*(\cdot, h))}{\hat{\phi}_{1:\ell-1|n}(L_{\ell-1,n}^*(\cdot, \mathbb{1}))} \right) \\ &\quad + \frac{\hat{\phi}_1(L_{1,n}^*(\cdot, h))}{\hat{\phi}_1(L_{1,n}^*(\cdot, \mathbb{1}))} - \frac{\phi_1^*(L_{1,n}^*(\cdot, h))}{\phi_1^*(L_{1,n}^*(\cdot, \mathbb{1}))}. \end{aligned} \quad (14)$$

This smoothing error can be written using filtering distributions only by introducing the following backward operators:

$$\begin{aligned} \mathcal{L}_{\ell,n}^*(x_\ell, h) &:= \sum_{x_{1:\ell-1}} B_{\phi_{\ell-1}^*}^*(x_\ell, x_{\ell-1}) \dots B_{\phi_1^*}^*(x_2, x_1) L_{\ell,n}^*(x_{1:\ell}, h), \\ \hat{\mathcal{L}}_{\ell,n}(x_\ell, h) &:= \sum_{x_{1:\ell-1}} \hat{B}_{\hat{\phi}_{\ell-1}}(x_\ell, x_{\ell-1}) \dots \hat{B}_{\hat{\phi}_1}(x_2, x_1) L_{\ell,n}^*(x_{1:\ell}, h), \end{aligned}$$

where for all $\nu \in \mathcal{P}(\mathcal{X})$, B_ν is the backward smoothing kernel given by

$$B_\nu^*(x, x') := \frac{\mathbf{Q}_*(x', x) \nu(x')}{\sum_{z \in \mathcal{X}} \mathbf{Q}_*(z, x) \nu(z)}.$$

Then, for all $2 \leq t \leq n$, the one step error at time ℓ is given by

$$\delta_{\ell,n}(h) := \frac{\hat{\phi}_{1:\ell|n}(L_{\ell,n}^*(\cdot, h))}{\hat{\phi}_{1:\ell|n}(L_{\ell,n}^*(\cdot, \mathbb{1}))} - \frac{\hat{\phi}_{1:\ell-1|n}(L_{\ell-1,n}^*(\cdot, h))}{\hat{\phi}_{1:\ell-1|n}(L_{\ell-1,n}^*(\cdot, \mathbb{1}))} = \frac{\hat{\phi}_\ell(\hat{\mathcal{L}}_{\ell,n}(\cdot, h))}{\hat{\phi}_\ell(\hat{\mathcal{L}}_{\ell,n}(\cdot, \mathbb{1}))} - \frac{\hat{\phi}_{\ell-1}(\hat{\mathcal{L}}_{\ell-1,n}(\cdot, h))}{\hat{\phi}_{\ell-1}(\hat{\mathcal{L}}_{\ell-1,n}(\cdot, \mathbb{1}))}. \quad (15)$$

This decomposition allows to obtain the upper bound for the marginal smoothing error stated in Proposition 2.2. The result is obtained by applying the decompositions (14) and (15) to a bounded function h on \mathcal{X}^n which depends on x_k only: for all $(x_1, \dots, x_n) \in \mathcal{X}^n$, $h(x_1, \dots, x_n) = h(x_k)$. The one step error given by (15) is then analyzed separately whether $k \geq \ell$ or $k < \ell$.

Case $k \geq \ell$

In this case, the function $L_{\ell,n}^*(\cdot, h)$ defined in (13) depends on x_ℓ only. Therefore, $\hat{\mathcal{L}}_{\ell,n}(x_\ell, h) = L_{\ell,n}^*(x_\ell, h) = \mathcal{L}_{\ell,n}^*(x_\ell, h)$. Thus, $\hat{\mathcal{L}}_{\ell-1,n}(x_{\ell-1}, h) = \sum_{x_\ell \in \mathcal{X}} \mathbf{Q}_*(x_{\ell-1}, x_\ell) f_{x_\ell}^*(y_\ell) \mathcal{L}_{\ell,n}^*(x_\ell, h)$ and the one step error given by (15) becomes

$$\delta_{\ell,n}(h) = \frac{\hat{\phi}_\ell(\mathcal{L}_{\ell,n}^*(\cdot, h))}{\hat{\phi}_\ell(\mathcal{L}_{\ell,n}^*(\cdot, \mathbb{1}))} - \frac{\hat{\phi}_{\ell-1}(\sum_{x_\ell \in \mathcal{X}} \mathbf{Q}_*(\cdot, x_\ell) f_{x_\ell}^*(y_\ell) \mathcal{L}_{\ell,n}^*(x_\ell, h))}{\hat{\phi}_{\ell-1}(\sum_{x_\ell \in \mathcal{X}} \mathbf{Q}_*(\cdot, x_\ell) f_{x_\ell}^*(y_\ell) \mathcal{L}_{\ell,n}^*(x_\ell, \mathbb{1}))}.$$

Define the measures μ_ℓ and $\hat{\mu}_\ell$ on \mathcal{X} by $\mu_\ell(x_\ell) := \sum_{x_{\ell-1} \in \mathcal{X}} \hat{\phi}_{\ell-1}(x_{\ell-1}) \mathbf{Q}_\star(x_{\ell-1}, x_\ell) f_{x_\ell}^\star(y_\ell)$ and $\hat{\mu}_\ell(x_\ell) := \sum_{x_{\ell-1} \in \mathcal{X}} \hat{\phi}_{\ell-1}(x_{\ell-1}) \hat{\mathbf{Q}}(x_{\ell-1}, x_\ell) \hat{f}_{x_\ell}(y_\ell)$. Then,

$$\delta_{\ell,n}(h) = \frac{\hat{\mu}_\ell(\mathcal{L}_{\ell,n}^\star(\cdot, h))}{\hat{\mu}_\ell(\mathcal{L}_{\ell,n}^\star(\cdot, \mathbb{1}))} - \frac{\mu_\ell(\mathcal{L}_{\ell,n}^\star(\cdot, h))}{\mu_\ell(\mathcal{L}_{\ell,n}^\star(\cdot, \mathbb{1}))}.$$

By [6, Lemma 4.3.23] and **[H1]-b)**, $|\delta_{\ell,n}(h)| \leq \rho_\star^{k-\ell} (1 - \delta^\star) \|\mu_\ell/\mu_\ell(\mathbb{1}) - \hat{\mu}_\ell/\hat{\mu}_\ell(\mathbb{1})\|_{\text{tv}} \|h\|_\infty / \delta^\star$. Following the same steps as for the proof of Proposition 2.1 yields

$$\|\mu_\ell/\mu_\ell(\mathbb{1}) - \hat{\mu}_\ell/\hat{\mu}_\ell(\mathbb{1})\|_{\text{tv}} \leq 2\|\mathbf{Q}_\star - \hat{\mathbf{Q}}\|_F / \delta^\star + 2c_\star^{-1}(y_\ell) \max_{x \in \mathcal{X}} |f_x^\star(y_\ell) - \hat{f}_x(y_\ell)|.$$

The term $\hat{\phi}_1(L_{1,n}^\star(\cdot, h))/\hat{\phi}_1(L_{1,n}^\star(\cdot, \mathbb{1})) - \phi_1^\star(L_{1,n}^\star(\cdot, h))/\phi_1^\star(L_{1,n}^\star(\cdot, \mathbb{1}))$ is dealt with similarly.

Case $k < \ell$

In this case, $L_{\ell,n}^\star(x_{1:\ell}, h) = h(x_k) \mathcal{L}_{\ell,n}^\star(x_\ell, \mathbb{1})$. Therefore,

$$\begin{aligned} \hat{\mathcal{L}}_{\ell,n}(x_\ell, h) &= \sum_{x_{1:\ell-1}} \hat{B}_{\hat{\phi}_{\ell-1}}(x_\ell, x_{\ell-1}) \dots \hat{B}_{\hat{\phi}_1}(x_2, x_1) h(x_k) \mathcal{L}_{\ell,n}^\star(x_\ell, \mathbb{1}), \\ &= \sum_{x_{k:\ell-1}} \mathcal{L}_{\ell,n}^\star(x_\ell, \mathbb{1}) \hat{B}_{\hat{\phi}_{\ell-1}}(x_\ell, x_{\ell-1}) \dots \hat{B}_{\hat{\phi}_k}(x_{k+1}, x_k) h(x_k). \end{aligned}$$

On the other hand, if $\nu_\ell(x_\ell) := \sum_{x_{\ell-1} \in \mathcal{X}} \hat{\phi}_{\ell-1}(x_{\ell-1}) \mathbf{Q}_\star(x_{\ell-1}, x_\ell) f_{x_\ell}^\star(y_\ell) \mathcal{L}_{\ell,n}^\star(x_\ell, \mathbb{1})$,

$$\hat{\phi}_{\ell-1}(\hat{\mathcal{L}}_{\ell-1,n}(\cdot, h)) = \sum_{x_{k:\ell} \in \mathcal{X}^{\ell-k+1}} \nu_\ell(x_\ell) \hat{B}_{\hat{\phi}_{\ell-1}}(x_\ell, x_{\ell-1}) \dots \hat{B}_{\hat{\phi}_k}(x_{k+1}, x_k) h(x_k).$$

Define $\hat{\nu}_\ell(x_\ell) := \hat{\phi}_\ell(x_\ell) \mathcal{L}_{\ell,n}^\star(x_\ell, \mathbb{1}) = \sum_{x_{\ell-1} \in \mathcal{X}} \hat{\phi}_{\ell-1}(x_{\ell-1}) \hat{\mathbf{Q}}(x_{\ell-1}, x_\ell) \hat{f}_{x_\ell}(y_\ell) \mathcal{L}_{\ell,n}^\star(x_\ell, \mathbb{1})$. Then, the one step error given by (15) becomes

$$\delta_{\ell,n}(h) = \sum_{x_{k:\ell-1}} \left(\frac{\hat{\nu}_\ell(x_\ell)}{\hat{\nu}_\ell(\mathbb{1})} - \frac{\nu_\ell(x_\ell)}{\nu_\ell(\mathbb{1})} \right) \hat{B}_{\hat{\phi}_{\ell-1}}(x_\ell, x_{\ell-1}) \dots \hat{B}_{\hat{\phi}_k}(x_{k+1}, x_k) h(x_k)$$

By [6, Lemma 4.3.23] and the fact that, for all $(x, x') \in \mathcal{X}^2$, $\hat{\mathbf{Q}}(x, x') \geq \hat{\delta}$,

$$|\delta_{\ell,n}(h)| \leq \|h\|_\infty \hat{\rho}^{\ell-k} \left\| \frac{\hat{\nu}_\ell(\cdot)}{\hat{\nu}_\ell(\mathbb{1})} - \frac{\nu_\ell(\cdot)}{\nu_\ell(\mathbb{1})} \right\|_{\text{tv}}.$$

As for all $x_\ell \in \mathcal{X}$, $\mathcal{L}_{\ell,n}^\star(x_\ell, \mathbb{1})/\|\mathcal{L}_{\ell,n}^\star(\cdot, \mathbb{1})\|_\infty \geq \delta^\star/(1 - \delta^\star)$, following the same steps as for the proof of Proposition 2.1 yields

$$\left\| \frac{\hat{\nu}_\ell(\cdot)}{\hat{\nu}_\ell(\mathbb{1})} - \frac{\nu_\ell(\cdot)}{\nu_\ell(\mathbb{1})} \right\|_{\text{tv}} \leq 2 \left(\frac{1 - \delta^\star}{\delta^\star} \right) \left(\|\mathbf{Q}_\star - \hat{\mathbf{Q}}\|_F / \delta^\star + c_\star^{-1}(y_\ell) \max_{x \in \mathcal{X}} |f_x^\star(y_\ell) - \hat{f}_x(y_\ell)| \right).$$

C Nonparametric spectral estimators

Theorem 3.1 follows from the following more precise results proved in this section. The proofs of the intermediate lemmas require assumptions **[H1']** and **[H2]-[H3]**.

Lemma C.1. *There exist a constant $0 < \sigma_{K, \mathfrak{F}^\star} \leq 1$ and a positive integer $M_{\mathfrak{F}^\star}$ such that for all $M \geq M_{\mathfrak{F}^\star}$,*

$$\sigma_K(\mathbf{O}_M) \geq \sigma_{K, \mathfrak{F}^\star} > 0.$$

Proof. By **[H3]**, the $(K \times K)$ Gram matrix defined by $\mathbf{O}_\star^\top \mathbf{O}_\star := (\langle f_{x_1}^\star, f_{x_2}^\star \rangle)_{x_1, x_2 \in \mathcal{X}}$ is invertible. Let $\varepsilon_{\mathfrak{F}^\star, M}$ be given by:

$$\varepsilon_{\mathfrak{F}^\star, M} := \|\mathbf{O}_M^\top \mathbf{O}_M - \mathbf{O}_\star^\top \mathbf{O}_\star\| = \|(\langle f_{M, x_1}^\star, f_{M, x_2}^\star \rangle - \langle f_{x_1}^\star, f_{x_2}^\star \rangle)_{x_1, x_2 \in \mathcal{X}}\|. \quad (16)$$

From (5), there exists $M_{\mathfrak{F}^\star} \geq 1$ such that for all $M \geq M_{\mathfrak{F}^\star}$, $\varepsilon_{\mathfrak{F}^\star, M} \leq 3\lambda_K(\mathbf{O}_\star^\top \mathbf{O}_\star)/4$. By Weyl's inequality (see Theorem D.1), $\sigma_K^2(\mathbf{O}_M) = \lambda_k(\mathbf{O}_M^\top \mathbf{O}_M) \geq \lambda_K(\mathbf{O}_\star^\top \mathbf{O}_\star)/4$. If $\sigma_K(\mathbf{O}_\star) := \lambda_K^{1/2}(\mathbf{O}_\star^\top \mathbf{O}_\star)$, note that for all $M \geq M_{\mathfrak{F}^\star}$, $\sigma_K(\mathbf{O}_M) \geq \sigma_K(\mathbf{O}_\star)/2$, which concludes the proof. \square

Define the *pseudo spectral gap* \mathbb{G}_{ps} of the Markov chain $(X_n)_{n \geq 1}$ as

$$\mathbb{G}_{\text{ps}} := \max_{k \geq 1} \left\{ \mathbb{G} \left(\text{Diag}[\pi^*]^{-1} (\mathbf{Q}_*^\top)^k \text{Diag}[\pi^*] \mathbf{Q}_*^k \right) / k \right\},$$

where $\mathbb{G}(A)$ denotes the spectral gap of a transition matrix A defined by

$$\mathbb{G}(A) := \begin{cases} 1 - \max\{\lambda : \lambda \text{ eigenvalue of } A, \lambda \neq 1\} & \text{if eigenvalue 1 has multiplicity 1,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that \mathbb{G}_{ps} depends only on the transition matrix \mathbf{Q}_* which is assumed to be aperiodic and irreducible with unique stationary distribution π^* . Perron-Frobenius theorem ensures that the spectral gap $\mathbb{G}(A)$ is well defined and such that $0 \leq \mathbb{G}(A) \leq 2$.

Remark C.1. If \mathbf{Q}_* is aperiodic and irreducible then $\mathbb{G}_{\text{ps}} > 0$. In this case, there exists k such that \mathbf{Q}_*^k is positive (entrywise) and so is $A := \text{Diag}[\pi^*]^{-1} (\mathbf{Q}_*^\top)^k \text{Diag}[\pi^*] \mathbf{Q}_*^k$. As A is a positive transition matrix, Perron-Frobenius theorem ensures that its spectral gap is positive.

Remark C.2. If \mathbf{Q}_* is aperiodic, irreducible and reversible then $\mathbb{G}_{\text{ps}} = \mathbb{G}(\mathbf{Q}_*)(2 - \mathbb{G}(\mathbf{Q}_*)) > 0$, see [24] and references therein.

Define the mixing time \mathbb{T}_{mix} of the Markov chain $(X_n)_{n \geq 1}$ as

$$\mathbb{T}_{\text{mix}} := \frac{1 + 3 \log 2 - \log \pi_{\min}^*}{\mathbb{G}_{\text{ps}}}.$$

This mixing time has a deeper interpretation in terms of convergence towards the stationary distribution in total variation norm, see [24] for instance. For any $\delta \in (0, 1)$, set

$$\mathcal{C}_*(\mathbf{Q}_*, \delta) := \sqrt{2/\mathbb{G}_{\text{ps}}} + 2\sqrt{-2\mathbb{T}_{\text{mix}} \log \delta}, \quad (17)$$

which is a constant that depends only on \mathbf{Q}_* and δ .

Theorem C.2. Assume that [H1'] and [H2]-[H3] hold. Let $\delta, \delta' \in (0, 1)$ then, with probability greater than $1 - 2\delta - 4\delta'$, there exists a permutation $\tau \in S_K$ such that the spectral method estimators $\hat{f}_{M,x}$, $\hat{\pi}$ and $\hat{\mathbf{Q}}$ (see Algorithm 1 for a definition) satisfy, for any $M \geq M_{\mathfrak{F}^*}$,

- for all $p \geq \mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')$ and all $x \in \mathcal{X}$,

$$\|f_{M,x}^* - \hat{f}_{M,\tau(x)}\|_2 \leq \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \eta_3(\Phi_M) / \sqrt{p}, \quad (18)$$

- for all $p \geq \mathbf{N}_2(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')$,

$$\|\mathbf{Q}_* - \mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top\| \leq \mathcal{D}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \eta_3(\Phi_M) / \sqrt{p}, \quad (19)$$

- for all $p \geq \mathbf{N}_3(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')$,

$$\|\pi^* - \mathbb{P}_\tau \hat{\pi}\|_2 \leq \mathcal{E}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \eta_3(\Phi_M) / \sqrt{p}, \quad (20)$$

where \mathbb{P}_τ is the permutation matrix associated to τ , and

$$\begin{aligned} \mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta') &:= \frac{4K}{3\sigma_{K,\mathfrak{F}^*}^2} \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)^2 \mathcal{C}_*(\mathbf{Q}_*, \delta')^2 \eta_3(\Phi_M)^2, \\ \mathbf{N}_2(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta') &:= \frac{4}{\pi_{\min}^{*2}} \mathcal{D}'_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)^2 \mathcal{C}_*(\mathbf{Q}_*, \delta')^2 \eta_3(\Phi_M)^2, \\ \mathbf{N}_3(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta') &:= \frac{4}{\sigma_K^2(\mathbf{A}_{\mathbf{Q}_*})} \mathcal{D}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)^2 \mathcal{C}_*(\mathbf{Q}_*, \delta')^2 \eta_3(\Phi_M)^2, \end{aligned}$$

with

$$\begin{aligned}
\mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) &:= \frac{2}{\sqrt{M}} \frac{\max_{x \in \mathcal{X}} \|f_x^*\|_2}{\sigma_{K, \mathfrak{F}^*}^2 \pi_{\min}^* \sigma_K(\mathbf{Q}_*^2)} + \left[1 + \frac{\|g^*\|_2}{\pi_{\min}^* \sigma_{K, \mathfrak{F}^*}^2 \sigma_K(\mathbf{Q}_*^2)} \frac{1}{\sqrt{M}} \right] \\
&\quad \times \left[\frac{13 \kappa^2(\mathbf{Q}_*) K^{1/2}}{\pi_{\min}^* \sigma_K(\mathbf{Q}_*^2)} \frac{\kappa_{\mathfrak{F}^*}^2}{\sigma_{K, \mathfrak{F}^*}^2} + \frac{83}{\delta} \frac{\kappa^6(\mathbf{Q}_*) K^{-5}}{\pi_{\min}^* \sigma_K(\mathbf{Q}_*^2)} \frac{\kappa_{\mathfrak{F}^*}^6 \max_{k \in \mathcal{X}} \|f_k^*\|_2}{\sigma_{K, \mathfrak{F}^*}^3} \left\{ 1 + \left(2 \log \frac{K^2}{\delta} \right)^{1/2} \right\} \right], \\
\mathcal{D}'_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) &:= \frac{2}{3 \sigma_{K, \mathfrak{F}^*}^2} \left[4 \sqrt{K} \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \max_{x \in \mathcal{X}} \|f_x^*\|_2 + \frac{3 \sqrt{3} \sigma_{K, \mathfrak{F}^*}}{M} \right], \\
\mathcal{D}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) &:= \frac{8 \|f_{(Y_1, Y_3)}^*\|_2}{3 \sigma_{K, \mathfrak{F}^*}^2 \pi_{\min}^{*2}} \left[\mathcal{D}'_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) + 4 \sqrt{3K} \pi_{\min}^* \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) + \frac{5 \pi_{\min}^*}{\|f_{(Y_1, Y_3)}^*\|_2 \sqrt{M}} \right], \\
\mathcal{E}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) &:= \frac{16 \|f_{(Y_1, Y_3)}^*\|_2}{\sigma_K^2(\mathbf{A}_{\mathbf{Q}_*}) \sigma_{K, \mathfrak{F}^*}^2 \pi_{\min}^{*2}} \left[\mathcal{D}'_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) + 4 \sqrt{3K} \pi_{\min}^* \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) + \frac{5 \pi_{\min}^*}{\|f_{(Y_1, Y_3)}^*\|_2 \sqrt{M}} \right],
\end{aligned}$$

where $\kappa_{\mathfrak{F}^*}$ is given in Lemma C.4, for all $(y_1, y_2, y_3) \in \mathcal{Y}^3$,

$$g^*(y_1, y_2, y_3) := \sum_{x_1, x_2, x_3 \in \mathcal{X}} \pi^*(x_1) \mathbf{Q}_*(x_1, x_2) \mathbf{Q}_*(x_2, x_3) f_{x_1}^*(y_1) f_{x_2}^*(y_2) f_{x_3}^*(y_3),$$

and $\sigma_K^2(\mathbf{A}_{\mathbf{Q}_*})$ is the K -th largest singular value of $\begin{pmatrix} \text{Id}_K - (\mathbf{Q}_*)^\top \\ \mathbf{1}_K^\top \end{pmatrix}$ (which is positive, see (29)).

Theorem C.2 is proved using the analysis of [3] to control the L^2 -error of the estimation based on the spectral method described in Section 3.1. To use their result in the nonparametric framework, it is essential to state explicitly how all constants depend on the dimension M . We thus need to recast and optimize the results of [3]. This is done in Theorem C.3 which is proved in Appendix F. Define

$$\gamma(\mathbf{O}_M) := \min_{x_1 \neq x_2} \|\mathbf{O}_M(\cdot, x_1) - \mathbf{O}_M(\cdot, x_2)\|_2 \quad (21)$$

and for all $A \in \mathbb{R}^{M \times M \times M}$ and all $B \in \mathbb{R}^{M \times K}$

$$\|A\|_{\infty, 2} := \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b A(\cdot, b, \cdot) \right\| \quad \text{and} \quad \|B\|_{2, \infty} := \max_{x \in \mathcal{X}} \|B(\cdot, x)\|_2. \quad (22)$$

Theorem C.3. Let $0 < \delta < 1$. Assume that $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ and that

$$8.2K^{5/2}(K-1) \frac{\kappa^2(\mathbf{Q}_* \mathbf{O}_M^\top)}{\delta \gamma(\mathbf{O}_M) \sigma_K(\mathbf{P}_M)} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty, 2} + \frac{\|\mathbf{M}_M\|_{\infty, 2} \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] < 1, \quad (23)$$

$$43.4K^4(K-1) \frac{\kappa^4(\mathbf{Q}_* \mathbf{O}_M^\top)}{\delta \gamma(\mathbf{O}_M) \sigma_K(\mathbf{P}_M)} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty, 2} + \frac{\|\mathbf{M}_M\|_{\infty, 2} \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] \leq 1, \quad (24)$$

then, with probability greater than $1 - 2\delta$, the matrix $\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}$ is invertible, the random matrix $\hat{\mathbf{C}}(1)$ is diagonalisable (see Algorithm 1), and there exists a permutation $\tau \in S_K$ such that for all $x \in \mathcal{X}$,

$$\begin{aligned}
\|\mathbf{O}_M(\cdot, x) - \hat{\mathbf{O}}_M(\cdot, \tau(x))\|_2 &\leq \frac{2\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \|\mathbf{O}_M\|_{2, \infty} + \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty, 2} + \frac{\|\mathbf{M}_M\|_{\infty, 2} \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] \\
&\quad \times \left[13K^{1/2} \frac{\kappa^2(\mathbf{Q}_* \mathbf{O}_M^\top)}{\sigma_K(\mathbf{P}_M)} + 116K^5 \left\{ 1 + (2 \log(K^2/\delta))^{1/2} \right\} \frac{\kappa^6(\mathbf{Q}_* \mathbf{O}_M^\top) \|\mathbf{O}_M\|_{2, \infty}}{\delta \gamma(\mathbf{O}_M) \sigma_K(\mathbf{P}_M)} \right].
\end{aligned}$$

Preliminary lemmas

Lemma C.4. There exists a constant $\kappa_{\mathfrak{F}^*}$ that depends only on \mathfrak{F}^* such that for all $M \geq M_{\mathfrak{F}^*}$, $\kappa(\mathbf{O}_M) \leq \kappa_{\mathfrak{F}^*}$ where $M_{\mathfrak{F}^*}$ is given in Lemma C.1. For all $M \geq M_{\mathfrak{F}^*}$, $\kappa(\mathbf{Q}_* \mathbf{O}_M^\top) \leq \kappa_{\mathfrak{F}^*} \kappa(\mathbf{Q}_*)$.

Proof. Note that $\mathbf{O}_*^\top \mathbf{O}_*$ is nonsingular. From (5) and (16) we deduce that $\mathbf{O}_M^\top \mathbf{O}_M$ tends to $\mathbf{O}_*^\top \mathbf{O}_*$ as M grows to infinity. This proves the first point. Recall that $\sigma_i(AB) \leq \sigma_1(A)\sigma_i(B)$ for all $i = 1, \dots, K$. Applying this identity to $A = \mathbf{Q}_*^{-1}$ and $B = \mathbf{Q}_* \mathbf{O}_M^\top$ yields $\sigma_K(\mathbf{Q}_*)\sigma_K(\mathbf{O}_M) \leq \sigma_K(\mathbf{Q}_* \mathbf{O}_M^\top)$. It follows that $\kappa(\mathbf{Q}_* \mathbf{O}_M^\top) \leq \kappa(\mathbf{Q}_*)\kappa(\mathbf{O}_M)$. The second claim follows from the first claim. \square

Lemma C.5. For all $M \geq M_{\mathfrak{F}^*}$, $\gamma(\mathbf{O}_M) \geq \sqrt{2}\sigma_{K,\mathfrak{F}^*}$ and $\|\mathbf{O}_M\|_{2,\infty} \leq \max_{x \in \mathcal{X}} \|f_x^*\|_2$, where $\gamma(\mathbf{O}_M)$ and $\|\mathbf{O}_M\|_{2,\infty}$ are defined in (21) and (22).

Proof. Observe that $\|\mathbf{O}_M v\|_2 \geq \sigma_K(\mathbf{O}_M)\|v\|_2$. With an appropriate choice of v and using Lemma C.1 this proves the first inequality. As Φ_M is an orthonormal family, $\|\mathbf{O}_M(\cdot, x)\|_2 \leq \|f_x^*\|_2$ which proves the second claim. \square

Lemma C.6. For all $M \geq 1$,

$$\|\mathbf{M}_M\|_{\infty,2} := \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b \mathbf{M}_M(\cdot, b, \cdot) \right\| \leq \|g^*\|_2,$$

where $\|\cdot\|_{\infty,2}$ is defined in (22).

Proof. As for all $x \in \mathcal{X}$, $f_x^* \in L^2(\mathcal{Y}, \mathcal{L}^D)$, $g^* \in L^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})$. Denote by $\langle \cdot, \cdot \rangle_{L^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})}$ the inner product of $L^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})$. As $\varphi_{a,b,c}(y_1, y_2, y_3) := \varphi_a(y_1)\varphi_b(y_2)\varphi_c(y_3)$ is an orthonormal family of $L^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})$,

$$\begin{aligned} \|\mathbf{M}_M\|_{\infty,2} &= \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b \mathbf{M}_M(\cdot, b, \cdot) \right\| \leq \max_{\|v\|_2=1} \sum_{b=1}^M |v_b| \|\mathbf{M}_M(\cdot, b, \cdot)\|, \\ &\leq \left(\sum_{b=1}^M \|\mathbf{M}_M(\cdot, b, \cdot)\|^2 \right)^{1/2} \leq \left(\sum_{b=1}^M \|\mathbf{M}_M(\cdot, b, \cdot)\|_F^2 \right)^{1/2}, \\ &= \left(\sum_{a,b,c=1}^M \mathbb{E} [\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)]^2 \right)^{1/2} = \left(\sum_{a,b,c=1}^M \langle g^*, \varphi_{a,b,c} \rangle_{L^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})}^2 \right)^{1/2} \leq \|g^*\|_2. \end{aligned}$$

using Cauchy-Schwarz inequality. \square

Lemma C.7. For all $M \geq 1$, $\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} \leq \|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F$, where $\|\cdot\|_{\infty,2}$ is defined in (22).

Proof. For all $M \geq 1$,

$$\begin{aligned} \|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} &= \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b (\widehat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot) \right\| \leq \max_{\|v\|_2=1} \sum_{b=1}^M |v_b| \|(\widehat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot)\|, \\ &\leq \left(\sum_{b=1}^M \|(\widehat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot)\|^2 \right)^{1/2} \leq \left(\sum_{b=1}^M \|(\widehat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot)\|_F^2 \right)^{1/2}, \\ &= \|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F. \end{aligned}$$

using Cauchy-Schwarz inequality. \square

Lemma C.8. Under [H1'] and [H2], for all $M \geq 1$, $\sigma_K(\mathbf{P}_M) \geq \pi_{\min} \sigma_K^2(\mathbf{O}_M) \sigma_K(\mathbf{Q}^2)$. If [H3] holds, then, for all $M \geq M_{\mathfrak{F}^*}$,

$$\sigma_K(\mathbf{P}_M) \geq \sigma_{K,\mathfrak{F}^*}^2 \pi_{\min}^* \sigma_K(\mathbf{Q}_*^2),$$

where $M_{\mathfrak{F}^*}$ and $\sigma_{K,\mathfrak{F}^*}$ are defined in Lemma C.1.

Proof. By Lemma F.1 and (7),

$$\begin{aligned} \sigma_K(\mathbf{P}_M) &= \sigma_K(\mathbf{U}^\top \mathbf{P}_M \mathbf{U}) = \sigma_K((\mathbf{U}^\top \mathbf{O}_M) \mathfrak{D} \text{diag}[\pi^*] \mathbf{Q}_*^2 (\mathbf{U}^\top \mathbf{O}_M)^\top), \\ &\geq \sigma_K(\mathbf{U}^\top \mathbf{O}_M) \sigma_K(\mathfrak{D} \text{diag}[\pi^*] \mathbf{Q}_*^2 (\mathbf{U}^\top \mathbf{O}_M)^\top), \\ &= \sigma_K(\mathbf{O}_M) \sigma_K(\mathfrak{D} \text{diag}[\pi^*] \mathbf{Q}_*^2 (\mathbf{U}^\top \mathbf{O}_M)^\top), \\ &\geq \sigma_K(\mathfrak{D} \text{diag}[\pi^*]) \sigma_K(\mathbf{O}_M) \sigma_K((\mathbf{U}^\top \mathbf{O}_M)^\top) \sigma_K(\mathbf{Q}_*^2), \\ &= \pi_{\min}^* \sigma_K^2(\mathbf{O}_M) \sigma_K(\mathbf{Q}_*^2), \end{aligned}$$

which concludes the proof. \square

First step: Estimation of the emission laws using a spectral method

Appendix E shows that:

$$\begin{aligned} \mathbb{P}\left[\|\widehat{\mathbf{L}}_M - \mathbf{L}_M\|_F \geq C_\star(\mathbf{Q}_\star, \delta')\eta_1(\Phi_M)/\sqrt{p}\right] &\leq \delta', \quad \mathbb{P}\left[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq C_\star(\mathbf{Q}_\star, \delta')\eta_3(\Phi_M)/\sqrt{p}\right] \leq \delta', \\ \mathbb{P}\left[\|\widehat{\mathbf{N}}_M - \mathbf{N}_M\|_F \geq C_\star(\mathbf{Q}_\star, \delta')\eta_2(\Phi_M)/\sqrt{p}\right] &\leq \delta', \quad \mathbb{P}\left[\|\widehat{\mathbf{P}}_M - \mathbf{P}_M\|_F \geq C_\star(\mathbf{Q}_\star, \delta')\eta_2(\Phi_M)/\sqrt{p}\right] \leq \delta'. \end{aligned}$$

Using the preliminary lemmas of Section C and the elementary fact that $M\eta_1(\Phi_M) \leq \sqrt{M}\eta_2(\Phi_M) \leq \eta_3(\Phi_M)$, deduce that (23) and (24) along with $3\|\widehat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ are satisfied when $M \geq M_{\mathfrak{F}^\star}$ and $p \geq N_0(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta')$ where:

$$N_0(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta') := \frac{942}{\delta^2} \frac{\kappa^8(\mathbf{Q}_\star) K^{10}}{\pi_{\min}^{\star 2} \sigma_K^2(\mathbf{Q}_\star^2)} \frac{\kappa_{\mathfrak{F}^\star}^8}{\sigma_{K, \mathfrak{F}^\star}^6} \left(1 + \frac{\|g^\star\|_2}{\pi_{\min}^\star \sigma_{K, \mathfrak{F}^\star}^2 \sigma_K(\mathbf{Q}_\star^2)} \frac{1}{\sqrt{M}}\right)^2 C_\star(\mathbf{Q}_\star, \delta')^2 \eta_3(\Phi_M)^2.$$

Using Theorem C.3, with probability greater than $1 - 2\delta - 4\delta'$, there exists a permutation τ satisfying for any $M \geq M_{\mathfrak{F}^\star}$, $p \geq N_0(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta')$ and $x \in \mathcal{X}$,

$$\|\mathbf{O}_M(\cdot, x) - \widehat{\mathbf{O}}_M(\cdot, \tau(x))\|_2 \leq C_M(\mathbf{Q}_\star, \mathfrak{F}^\star, \delta) C_\star(\mathbf{Q}_\star, \delta') \eta_3(\Phi_M)/\sqrt{p}.$$

This proves the first part of Theorem C.2.

Second step: Preliminary estimation of the stationary density using a spectral method

For sake of readability, assume that τ is the identity permutation. Observe that:

$$N_1(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta') \geq N_0(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta').$$

Recall $\tilde{\pi} := (\hat{\mathbf{U}}^\top \widehat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \widehat{\mathbf{L}}_M$ and $\pi^\star = (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \mathbf{L}_M$.

Lemma C.9. *With probability greater than $1 - 2\delta - 4\delta'$, if $p > N_1(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta')$ then,*

$$\begin{aligned} \|\tilde{\pi} - \pi^\star\|_2 &\leq \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^\star}} C_\star(\mathbf{Q}_\star, \delta') \frac{\eta_1(\Phi_M)}{\sqrt{p}} \\ &\quad + \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^\star}} \frac{\sqrt{N_1(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta')}}{\sqrt{p} - \sqrt{N_1(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta')}} \left(\max_{x \in \mathcal{X}} \|f_x^\star\|_2 + C_\star(\mathbf{Q}_\star, \delta') \frac{\eta_1(\Phi_M)}{\sqrt{p}} \right). \end{aligned}$$

Proof. Set $A = \hat{\mathbf{U}}^\top \mathbf{O}_M$, $\tilde{A} = \hat{\mathbf{U}}^\top \widehat{\mathbf{O}}_M$ and $B = \hat{\mathbf{U}}^\top (\mathbf{O}_M - \widehat{\mathbf{O}}_M)$. Then,

$$\|B\| \leq \|\mathbf{O}_M - \widehat{\mathbf{O}}_M\| \leq \|\mathbf{O}_M - \widehat{\mathbf{O}}_M\|_F \leq \sqrt{K} \max_{x \in \mathcal{X}} \|\mathbf{O}_M(\cdot, x) - \widehat{\mathbf{O}}_M(\cdot, x)\|_2,$$

which gives $\|B\| \leq \sqrt{K} C_M(\mathbf{Q}_\star, \mathfrak{F}^\star, \delta) C_\star(\mathbf{Q}_\star, \delta') \eta_3(\Phi_M)/\sqrt{p}$. Similarly, by claim (iii) of Lemma F.3:

$$\|A^{-1}B\| \leq \|A^{-1}\| \|B\| \leq \sigma_K^{-1}(A) \|B\| \leq \frac{2\sqrt{K} \max_{x \in \mathcal{X}} \|\mathbf{O}_M(\cdot, x) - \widehat{\mathbf{O}}_M(\cdot, x)\|_2}{\sqrt{3}\sigma_K(\mathbf{O}_M)},$$

so that

$$\|A^{-1}B\| \leq \frac{2\sqrt{K}}{\sqrt{3}\sigma_{K, \mathfrak{F}^\star}} C_M(\mathbf{Q}_\star, \mathfrak{F}^\star, \delta) C_\star(\mathbf{Q}_\star, \delta') \frac{\eta_3(\Phi_M)}{\sqrt{p}}.$$

Observe that the condition on p and M ensures that $\|A^{-1}B\| < 1$. Apply Theorem D.2 to get that:

$$\|(\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} - (\hat{\mathbf{U}}^\top \widehat{\mathbf{O}}_M)^{-1}\| \leq \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^\star}} \frac{\sqrt{N_1(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta')}}{\sqrt{p} - \sqrt{N_1(\mathbf{Q}_\star, \mathfrak{F}^\star, \Phi_M, \delta, \delta')}}. \quad (25)$$

Furthermore, using (25):

$$\begin{aligned}
\|\tilde{\pi} - \pi^*\|_2 &= \|(\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M - (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \mathbf{L}_M\|_2 \\
&= \|(\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M - (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M + (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M - (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \mathbf{L}_M\|_2 \\
&\leq \|(\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} - (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1}\| \|\hat{\mathbf{L}}_M\|_2 + \|A^{-1}\| \|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_2 \\
&\leq \frac{2}{\sqrt{3}\sigma_{K,\mathfrak{F}^*}} \left(\|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_2 + \frac{\sqrt{\mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')}}{\sqrt{p} - \sqrt{\mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')}} (\|\mathbf{L}_M\|_2 + \|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_2) \right).
\end{aligned}$$

Denote $f_{Y_1}^* = \sum_{x_1 \in \mathcal{X}} \pi(x_1) f_{k_1}^*(y_1)$ the density of Y_1 . Observe that:

$$\|\mathbf{L}_M\|_2 = \left(\sum_{a=1}^M \mathbb{E}[\varphi_a(Y_1)]^2 \right)^{1/2} = \left(\sum_{a=1}^M \langle f_{Y_1}^*, \varphi_a \rangle^2 \right)^{1/2} \leq \|f_{Y_1}^*\|_2 \leq \max_{x \in \mathcal{X}} \|f_x^*\|_2,$$

which concludes the proof. \square

This results allows to state that for all $p \geq 4\mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')$,

$$\|\pi^* - \mathbb{P}_\tau \tilde{\pi}\|_2 \leq \mathcal{D}'_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \eta_3(\Phi_M) / \sqrt{p}. \quad (26)$$

Third step: Estimation of the transition matrix using a spectral method

Denote $\tilde{\mathbf{Q}} := (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M \mathfrak{D} \text{diag}[\tilde{\pi}])^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{N}}_M \hat{\mathbf{U}} (\hat{\mathbf{O}}_M^\top \hat{\mathbf{U}})^{-1}$. Observe $\hat{\mathbf{Q}} = \Pi_{TM}(\tilde{\mathbf{Q}})$ and $\mathbf{Q}_* = \Pi_{TM}(\mathbf{Q}_*)$ and hence, by non-expansivity of the projection onto convex sets, $\|\hat{\mathbf{Q}} - \mathbf{Q}_*\|_F \leq \|\tilde{\mathbf{Q}} - \mathbf{Q}_*\|_F$. Moreover, notice that:

$$\mathbf{N}_2(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta') \geq 4\mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta') \geq \mathbf{N}_0(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta').$$

Lemma C.10. *With probability greater than $1 - 2\delta - 4\delta'$, if $p \geq \mathbf{N}_2(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')$ then*

$$\|\tilde{\mathbf{Q}} - \mathbf{Q}_*\| \leq \frac{8\|f_{(Y_1, Y_3)}^*\|_2}{3\sigma_{K,\mathfrak{F}^*}^2 \pi_{\min}^*} \|\tilde{\pi} - \pi^*\|_2 + \frac{2}{\pi_{\min}^*} \tilde{\mathcal{E}}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \frac{\eta_3(\Phi_M)}{\sqrt{p}},$$

where

$$\tilde{\mathcal{E}}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) := \frac{16}{\sqrt{3}\sigma_{K,\mathfrak{F}^*}^2} \left[\sqrt{K} \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \|f_{(Y_1, Y_3)}^*\|_2 + \frac{5}{4\sqrt{3M}} \right].$$

Proof. Observe that (20) shows that $\|\tilde{\pi} - \pi^*\|_2 \leq \pi_{\min}^*/2$. Then, for any $x \in \mathcal{X}$:

$$\tilde{\pi}_x \geq \frac{\pi_{\min}^*}{2} > 0. \quad (27)$$

Set $\mathbf{V} = (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top$ and $\hat{\mathbf{V}} = (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top$. Note $\tilde{\mathbf{Q}} = \mathfrak{D} \text{diag}[\tilde{\pi}]^{-1} \hat{\mathbf{V}} \hat{\mathbf{N}}_M \hat{\mathbf{V}}^\top$ and:

$$\mathbf{Q} = \mathfrak{D} \text{diag}[\pi^*]^{-1} \mathbf{V} \mathbf{N}_M \mathbf{V}^\top.$$

Set $E = \hat{\mathbf{V}} - \mathbf{V}$ and $F = \hat{\mathbf{N}}_M - \mathbf{N}_M$. Using (25) yields:

$$\|E\| \leq \frac{2}{\sqrt{3}\sigma_{K,\mathfrak{F}^*}} \frac{\sqrt{\mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')}}{\sqrt{p} - \sqrt{\mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta')}} \leq \frac{8\sqrt{K}}{3\sigma_{K,\mathfrak{F}^*}^2} \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \frac{\eta_3(\Phi_M)}{\sqrt{p}}.$$

By claim (iii) of Lemma F.3, $\|\mathbf{V}\| \leq \sigma_K^{-1} (\hat{\mathbf{U}}^\top \mathbf{O}_M) \leq 2/(\sqrt{3}\sigma_{K,\mathfrak{F}^*})$. Furthermore, $\varphi_{a,c}(y_1, y_3) := \varphi_a(y_1) \varphi_c(y_3)$ is an orthonormal family of $L^2(\mathcal{Y}^2, \mathcal{L}^{D \otimes 2})$ and

$$\|\mathbf{N}_M\|_F = \left(\sum_{a,c=1}^M \mathbb{E}[\varphi_a(Y_1) \varphi_c(Y_3)]^2 \right)^{1/2} = \left(\sum_{a,c=1}^M \langle f_{(Y_1, Y_3)}^*, \varphi_{a,c} \rangle_{L^2(\mathcal{Y}^2, \mathcal{L}^{D \otimes 2})}^2 \right)^{1/2} \leq \|f_{(Y_1, Y_3)}^*\|_2.$$

Then,

$$\begin{aligned}
\|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| &= \|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - (\mathbf{V} + E)(\mathbf{N}_M + F)(\mathbf{V} + E)^\top\|, \\
&= \|\mathbf{V}\mathbf{N}_ME^\top + \mathbf{V}F\mathbf{V}^\top + \mathbf{V}FE^\top + E\mathbf{N}_M\mathbf{V}^\top + E\mathbf{N}_ME^\top + EF\mathbf{V}^\top + EFE^\top\|, \\
&\leq 2\|E\|\|\mathbf{V}\|\|\mathbf{N}_M\| + 2\|E\|\|\mathbf{V}\|\|F\| + \|E\|^2\|\mathbf{N}_M\| + \|\mathbf{V}\|^2\|F\| + \|E\|^2\|F\|,
\end{aligned}$$

yields

$$\begin{aligned}
\|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| &\leq \frac{32\sqrt{K}\mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}_*, \delta')\|f_{(Y_1, Y_3)}^*\|_2}{3\sqrt{3}\sigma_{K, \mathfrak{F}^*}^3} \left[1 + \frac{\mathcal{C}_*(\mathbf{Q}_*, \delta')}{\|f_{(Y_1, Y_3)}^*\|_2} \frac{\eta_3(\Phi_M)}{\sqrt{pM}} \right. \\
&\quad + \frac{2\sqrt{K}\mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}_*, \delta')}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}} \frac{\eta_3(\Phi_M)}{\sqrt{p}} \\
&\quad + \frac{\sqrt{3}\sigma_{K, \mathfrak{F}^*}}{4\mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)\|f_{(Y_1, Y_3)}^*\|_2\sqrt{K}} \frac{1}{\sqrt{M}} \\
&\quad \left. + \frac{2\sqrt{K}\mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}_*, \delta')^2}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}\|f_{(Y_1, Y_3)}^*\|_2} \frac{\eta_3^2(\Phi_M)}{p\sqrt{M}} \right] \frac{\eta_3(\Phi_M)}{\sqrt{p}}
\end{aligned}$$

$$\text{As } p \geq \mathbf{N}_2(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta') \geq 4\mathbf{N}_1(\mathbf{Q}_*, \mathfrak{F}^*, \Phi_M, \delta, \delta') = \frac{16K}{3\sigma_{K, \mathfrak{F}^*}^2} \mathcal{C}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta)^2 \mathcal{C}_*(\mathbf{Q}_*, \delta')^2 \eta_3(\Phi_M)^2,$$

$$\|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| \leq \tilde{\mathcal{E}}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \frac{\eta_3(\Phi_M)}{\sqrt{p}}. \quad (28)$$

Observe that:

$$\begin{aligned}
\|\mathbf{Q}_* - \tilde{\mathbf{Q}}\| &= \|(\mathfrak{D}\text{diag}[\pi^*]^{-1} - \mathfrak{D}\text{diag}[\hat{\pi}]^{-1})\mathbf{V}\mathbf{N}_M\mathbf{V}^\top + \mathfrak{D}\text{diag}[\hat{\pi}]^{-1}(\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top)\| \\
&\leq \|\mathfrak{D}\text{diag}[\pi^*]^{-1} - \mathfrak{D}\text{diag}[\hat{\pi}]^{-1}\|\|\mathbf{V}\|^2\|\mathbf{N}_M\| + \|\mathfrak{D}\text{diag}[\hat{\pi}]^{-1}\|\|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| \\
&\leq \frac{4\|f_{(Y_1, Y_3)}^*\|_2}{3\sigma_{K, \mathfrak{F}^*}^2} \max_{x \in \mathcal{X}}(\pi_x^{*-1} - \hat{\pi}_x^{-1}) + \max_{x \in \mathcal{X}} \hat{\pi}_x^{-1} \tilde{\mathcal{E}}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \frac{\eta_3(\Phi_M)}{\sqrt{p}} \\
&\leq \frac{8\|f_{(Y_1, Y_3)}^*\|_2}{3\sigma_{K, \mathfrak{F}^*}^2 \pi_{\min}^2} \|\hat{\pi} - \pi^*\|_2 + \frac{2}{\pi_{\min}^*} \tilde{\mathcal{E}}_M(\mathbf{Q}_*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}_*, \delta') \frac{\eta_3(\Phi_M)}{\sqrt{p}},
\end{aligned}$$

using (27) and (28). \square

Combining (26) and Lemma C.10 proves the second point of Theorem C.2.

Last step: Final estimation of the stationary distribution

By [H1'], we know that the transition matrix \mathbf{Q}_* is irreducible and aperiodic. Perron-Frobenius theorem shows that \mathbf{Q}_* has a unique stationary distribution π^* . More precisely,

- $\mathbb{R} \cdot \pi^* = \ker(\text{Id}_K - (\mathbf{Q}_*)^\top)$ so that $(\mathbb{R} \cdot \pi^*)^\perp = \text{range}(\text{Id}_K - \mathbf{Q}_*)$,
- and $\langle \pi^*, \mathbb{1}_K \rangle = 1$,

where $\mathbb{1}_K = (1, \dots, 1) \in \mathbb{R}^K$. We deduce $\mathbb{1}_K \notin \text{range}(\text{Id}_K - \mathbf{Q}_*)$ and

$$\text{Rank} \left(\begin{pmatrix} \text{Id}_K - (\mathbf{Q}_*)^\top \\ \mathbb{1}_K^\top \end{pmatrix} \right) = K. \quad (29)$$

Set

$$A = \begin{pmatrix} \text{Id}_K - \mathbf{Q}^\top \\ \mathbb{1}_K^\top \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} \text{Id}_K - (\mathbf{Q}_*)^\top \\ \mathbb{1}_K^\top \end{pmatrix}.$$

We first derive an upper bound on $\|A^+ - (A^*)^+\|$ where A^+ denotes the Moore-Penrose pseudo-inverse of A . Note that

$$A^+ - (A^*)^+ = (A^*)^+(A^* - A)A^+ - (A^*)^+(\text{Id}_{K+1} - AA^+). \quad (30)$$

The last term can be written as

$$(A^*)^+(\text{Id}_{K+1} - AA^+) = (A^*)^+(A^*(A^*)^+)(\text{Id}_{K+1} - AA^+) = (A^*)^+P_{\text{range}(A^*)}P_{\text{range}(A)^\perp},$$

where $P_{\text{range}(A^*)} = A^*(A^*)^+$ denotes the orthogonal projection onto $\text{range}(A^*)$ and $P_{\text{range}(A)^\perp} = \text{Id}_{K+1} - AA^+$ denotes the orthogonal projection onto the orthogonal of $\text{range}(A)$. Define

$$s(\mathbf{Q}_*) := \sigma_K(A^*). \quad (31)$$

Lemma C.11. *If $\|\mathbf{Q} - \mathbf{Q}_*\| \leq s(\mathbf{Q}_*)/2$ then $\text{Rank}(A) = \text{Rank}(A^*) = K$ and*

$$\|P_{\text{range}(A^*)}P_{\text{range}(A)^\perp}\| \leq \frac{2\|\mathbf{Q} - \mathbf{Q}_*\|}{s(\mathbf{Q}_*)}.$$

Proof. The first point follows from Weyl's inequality, see Theorem D.1. By [28],

$$\|P_{\text{range}(A^*)^\perp}P_{\text{range}(A)}\| = \|P_{\text{range}(A)^\perp}P_{\text{range}(A^*)}\|.$$

Moreover, since projections P are orthogonal $(P_{\text{range}(A)^\perp}P_{\text{range}(A^*)})^\top = P_{\text{range}(A^*)}P_{\text{range}(A)^\perp}$. Using notation of [28], one may notice that $\|\sin \theta(\text{range}(A), \text{range}(A^*))\| = \|P_{\text{range}(A^*)^\perp}P_{\text{range}(A)}\|$. By Wedin's theorem [28], if $\sigma_K(A) \geq s(\mathbf{Q}_*)/2$ then $\|\sin \theta(\text{range}(A), \text{range}(A^*))\| \leq \frac{2\|A - A^*\|}{\sigma_K(A^*)}$. We conclude using Weyl's inequality, see Theorem D.1. \square

Triangular inequality in (30) gives

$$\begin{aligned} \|A^+ - (A^*)^+\| &\leq \|(A^*)^+\| \|\mathbf{Q} - \mathbf{Q}_*\| \left(\|A^+\| + \frac{2}{\sigma_K(A^*)} \right), \\ &\leq \frac{\|\mathbf{Q} - \mathbf{Q}_*\|}{\sigma_K(A^*)} \left(\|A^+ - (A^*)^+\| + \frac{3}{\sigma_K(A^*)} \right), \end{aligned}$$

using that $\|(A^*)^+\| = 1/\sigma_K(A^*)$. Deduce that if $\|\mathbf{Q} - \mathbf{Q}_*\| \leq \sigma_K(A^*)/2$ then $\|A^+ - (A^*)^+\| \leq 6\|\mathbf{Q} - \mathbf{Q}_*\|/\sigma_K^2(A^*)$. From Weyl's inequality, if $\|\mathbf{Q} - \mathbf{Q}_*\| \leq \sigma_K(A^*)/2$ then $\sigma_K(A) \geq \sigma_K(A^*)/2$. $\text{Id}_K - \mathbf{Q}^\top$ has rank $K - 1$ and the eigenspace $\ker(\text{Id}_K - \mathbf{Q}^\top)$ has dimension 1. Thus, \mathbf{Q} is an irreducible and aperiodic transition matrix, and π is the unique solution to

$$\begin{pmatrix} \text{Id}_K - \mathbf{Q}^\top \\ \mathbf{1}_K^\top \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now $\|\pi - \pi^*\|_2 \leq \|A^+ - (A^*)^+\|$ and the last part of Theorem C.2 is proved.

D Matrix perturbation

We gather in this section some useful results in matrix perturbation theory. Proofs of the following theorem may be found in [26] for instance.

Theorem D.1 (Weyl's inequality). *Let A, B be $(p \times q)$ matrices with $p \geq q$ then, for all $i = 1, \dots, q$,*

$$|\sigma_i(A + B) - \sigma_i(A)| \leq \sigma_1(B).$$

Theorem D.2. *Let A, B be $(p \times p)$ matrices. If A is invertible and $\|A^{-1}B\| < 1$ then $\tilde{A} := A + B$ is invertible and*

$$\|\tilde{A}^{-1} - A^{-1}\| \leq \frac{\|B\|\|A^{-1}\|^2}{1 - \|A^{-1}B\|}.$$

Theorem D.3 (Bauer-Fike). *Let A, B be $(p \times p)$ matrices and $\tilde{A} := A + B$. Assume that A is diagonalizable, i.e. $X^{-1}AX = \Lambda$, where $\Lambda = \text{Diag}[(\lambda_1, \dots, \lambda_p)]$. Then,*

$$\text{sv}_A(\tilde{A}) \leq \kappa(X)\|B\|, \quad (32)$$

where $\text{sv}_A(\tilde{A}) := \max_j \min_i |\tilde{\lambda}_j - \lambda_i|$ and $\tilde{\lambda}_j$ denotes the eigenvalues of \tilde{A} .

Remark D.1. *Moreover, if the disks $\mathcal{D}_i := \{\xi : |\xi - \lambda_i| \leq \kappa(X)\|B\|\}$ are isolated from the others, then (32) holds with the matching distance $\text{md}(A, \tilde{A}) \leq \kappa(X)\|B\|$ where $\text{md}(A, \tilde{A}) := \min_{\tau \in \mathcal{S}_p} \max_i |\hat{\lambda}_{\tau(i)} - \lambda_i|$.*

Eventually, if Λ, \tilde{A} are real valued matrices then \tilde{A} has p distinct real eigenvalues.

E Concentration inequalities

Consider consecutive observations of the same hidden Markov chain $Z_s := (Y_s, Y_{s+1}, Y_{s+2})$ for $1 \leq s \leq p$,

Lemma E.1. *For any positive u , any M and any p :*

$$\begin{aligned} \mathbb{P} \left[\|\widehat{\mathbf{L}}_M - \mathbf{L}_M\|_F \geq \frac{\sqrt{2}\eta_1(\Phi_M)}{\sqrt{p\mathbb{G}_{\text{ps}}}} (1 + 2u\sqrt{1 + \log(8/\pi_{\min}^*)}) \right] &\leq \exp(-u^2), \\ \mathbb{P} \left[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq \frac{\sqrt{2}\eta_3(\Phi_M)}{\sqrt{p\mathbb{G}_{\text{ps}}}} (1 + 2u\sqrt{1 + \log(8/\pi_{\min}^*)}) \right] &\leq \exp(-u^2), \\ \mathbb{P} \left[\|\widehat{\mathbf{N}}_M - \mathbf{N}_M\|_F \geq \frac{\sqrt{2}\eta_2(\Phi_M)}{\sqrt{p\mathbb{G}_{\text{ps}}}} (1 + 2u\sqrt{1 + \log(8/\pi_{\min}^*)}) \right] &\leq \exp(-u^2), \\ \mathbb{P} \left[\|\widehat{\mathbf{P}}_M - \mathbf{P}_M\|_F \geq \frac{\sqrt{2}\eta_2(\Phi_M)}{\sqrt{p\mathbb{G}_{\text{ps}}}} (1 + 2u\sqrt{1 + \log(8/\pi_{\min}^*)}) \right] &\leq \exp(-u^2). \end{aligned}$$

Proof. Set $\zeta_{\mathbf{L}_M}(Z_1, \dots, Z_p) := \|\widehat{\mathbf{L}}_M(Z_1, \dots, Z_p) - \mathbf{L}_M\|_2$, $\zeta_{\mathbf{M}_M}(Z_1, \dots, Z_p) := \|\widehat{\mathbf{M}}_M(Z_1, \dots, Z_p) - \mathbf{M}_M\|_F$, $\zeta_{\mathbf{N}_M}(Z_1, \dots, Z_p) := \|\widehat{\mathbf{N}}_M(Z_1, \dots, Z_p) - \mathbf{N}_M\|_F$ and $\zeta_{\mathbf{P}_M}(Z_1, \dots, Z_p) := \|\widehat{\mathbf{P}}_M(Z_1, \dots, Z_p) - \mathbf{P}_M\|_F$ where, for instance $\widehat{\mathbf{L}}_M(Z_1, \dots, Z_p)$ denotes the dependence of $\widehat{\mathbf{L}}_M$ in Z_1, \dots, Z_p . We begin with $\zeta_{\mathbf{M}_M}$, other cases are similar. Form the difference with respect to the coordinate i :

$$c_i := \sup_{z_j \in \mathcal{Y}^3, z'_i \in \mathcal{Y}^3} |\zeta_{\mathbf{M}_M}(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_p) - \zeta_{\mathbf{M}_M}(z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_p)|.$$

By the triangular inequality,

$$c_i \leq \sup_{z_j \in \mathcal{Y}^3, z'_i \in \mathcal{Y}^3} \left\| \widehat{\mathbf{M}}_M(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_p) - \widehat{\mathbf{M}}_M(z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_p) \right\|_F,$$

so that

$$c_i \leq \frac{1}{p} \sup_{z_i \in \mathcal{Y}^3, z'_i \in \mathcal{Y}^3} \left(\sum_{a,b,c} \left(\varphi_a(y_1^{(i)}) \varphi_b(y_2^{(i)}) \varphi_c(y_3^{(i)}) - \varphi_a(y_1'^{(i)}) \varphi_b(y_2'^{(i)}) \varphi_c(y_3'^{(i)}) \right)^2 \right)^{1/2}.$$

Eventually, we get that $c_i \leq \eta_3(\Phi_M)/p$. By McDiarmid's inequality [24], for all $u > 0$,

$$\mathbb{P}(\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq \mathbb{E}[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F] + u) \leq \exp\left(-\frac{pu^2}{8\mathbb{T}_{\text{mix}}\eta_3^2(\Phi_M)}\right).$$

We need the following lemma that can be deduced from [24].

Lemma E.2. *For any $a, b, c \in \{1, \dots, M\}$,*

$$\begin{aligned} \mathbb{E} \left[\sum_{s=1}^p \frac{1}{p} [\varphi_a(Y_s) \varphi_b(Y_{s+1}) \varphi_c(Y_{s+2}) - \mathbb{E}[\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3)]] \right]^2 \\ \leq \frac{4}{p\mathbb{G}_{\text{ps}}} \mathbb{E} [\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3) - \mathbb{E}[\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3)]]^2. \end{aligned}$$

Proof. Notice that $(X_1, Y_1), (X_2, Y_2), \dots$ is homogenous, irreducible, aperiodic and stationary Markov chain on $\mathcal{X} \times \mathcal{Y}$, whose stationary distribution is $\tilde{\pi}(x, dy) := \pi_x \mu_x(dy)$. Observe that its transition kernel $\tilde{\mathbf{Q}}$ satisfies, for all $x, x' \in \mathcal{X}$ and all $y, y' \in \mathcal{Y}$,

$$\tilde{\mathbf{Q}}(x, y; x', dy') = \mathbf{Q}_*(x, x') \mu_{x'}(dy').$$

The transition kernel $\tilde{\mathbf{Q}}$ can be viewed as an operator \mathbb{Q} on the Hilbert space $L^2(\tilde{\pi})$ defined, for all $f \in L^2(\tilde{\pi})$, by:

$$(\mathbb{Q}f)(x, y) := \mathbb{E}_{\tilde{\mathbf{Q}}(x, y; \cdot, \cdot)}(f) = \sum_{x' \in \mathcal{X}} \mathbf{Q}_*(x, x') \int_{\mathcal{Y}} f(x', y') \mu_{x'}(dy').$$

Note that $\mathbb{Q}f(x, y)$ does not depend on y . Set $E := \{f(x, y) \in L^2(\tilde{\pi}) : f \text{ does not depend on } y\}$. The $L^2(\tilde{\pi})$ -self-adjoint operator defined, for all $f \in L^2(\tilde{\pi})$, by

$$(\Pi_E f)(x, y) := \int_{\mathcal{Y}} f(x, y') \mu_x(dy'),$$

is the orthogonal projection onto E . Since $\Pi_E \mathbb{Q} \Pi_E = \mathbb{Q}$, the set of nonzero eigenvalues of \mathbb{Q} is exactly the set of nonzero eigenvalues of the K dimensional linear operator $\Pi_E \mathbb{Q} \Pi_E$. Eventually, note that the matrix of \mathbb{Q} in the basis $((x, y) \mapsto \mathbf{1}_{x'=x})_{x' \in \mathcal{X}}$ is \mathbf{Q}_\star . Then, the pseudo spectral gap of \mathbb{Q} is equal to \mathbb{G}_{ps} (the pseudo spectral gap of \mathbf{Q}_\star).

Furthermore, note the same analysis can be made for $(X_1, X_2, X_3, Z_1), (X_2, X_3, X_4, Z_2), \dots$ and its pseudo spectral gap is the pseudo spectral gap of the Markov chain $(X_1, X_2, X_3), (X_2, X_3, X_4), \dots$ which is \mathbb{G}_{ps} . Indeed, the set of nonzero eigenvalues of the Markov chain $(X_1, X_2, X_3), (X_2, X_3, X_4), \dots$ is equal to the set of nonzero eigenvalues of the Markov chain X_1, X_2, \dots .

Eventually, set $g(X_s, X_{s+1}, X_{s+2}, Z_s) := (1/p)\varphi_a(Y_s)\varphi_b(Y_{s+1})\varphi_c(Y_{s+2})$ and apply Theorem 3.7 in [24] to conclude the proof. \square

Then,

$$\begin{aligned} \mathbb{E} \left[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F \right] &\leq \mathbb{E} \left[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_F^2 \right]^{1/2}, \\ &\leq \mathbb{E} \left[\sum_{a,b,c} \left(\frac{1}{p} \sum_{s=1}^p \varphi_a(Y_s)\varphi_b(Y_{s+1})\varphi_c(Y_{s+2}) - \mathbb{E}[\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)] \right)^2 \right]^{1/2}, \\ &\leq \left[\sum_{a,b,c} \mathbb{E} \left(\sum_{s=1}^p \frac{1}{p} \{ \varphi_a(Y_s)\varphi_b(Y_{s+1})\varphi_c(Y_{s+2}) - \mathbb{E}[\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)] \} \right)^2 \right]^{1/2}, \\ &\leq \frac{2}{\sqrt{p\mathbb{G}_{\text{ps}}}} \left[\sum_{a,b,c} \mathbb{E} [\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3) - \mathbb{E}\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)]^2 \right]^{1/2}, \\ &\leq \left(\frac{2}{p\mathbb{G}_{\text{ps}}} \right)^{1/2} \left[\mathbb{E} \sum_{a,b,c} (\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3) - \varphi_a(Y'_1)\varphi_b(Y'_2)\varphi_c(Y'_3))^2 \right]^{1/2}, \\ &\leq \left(\frac{2\eta_3^2(\Phi_M)}{p\mathbb{G}_{\text{ps}}} \right)^{1/2}, \end{aligned}$$

using Jensen's inequality, Lemma E.2 and then $2\mathbb{E}(U - \mathbb{E}U)^2 \leq \mathbb{E}(U - U')^2$ where U is any real valued random variable with finite second moment and U' an independent copy of U . The proof is similar for \mathbf{L}_M , \mathbf{N}_M and \mathbf{P}_M . \square

F Proof of Theorem C.3

Preliminaries lemmas

Lemma F.1. For all $b \in \{1, \dots, M\}$,

$$\mathbf{M}_M(\cdot, b, \cdot) = \mathbf{O}_M \mathfrak{Diag}[\pi^\star] \mathbf{Q}_\star \mathfrak{Diag}[\mathbf{O}_M(b, \cdot)] \mathbf{Q}_\star \mathbf{O}_M^\top.$$

Similarly, $\mathbf{P}_M = \mathbf{O}_M \mathfrak{Diag}[\pi^\star] \mathbf{Q}_\star^2 \mathbf{O}_M^\top$.

Proof. Let $a, c \in \{1, \dots, M\}^2$ and observe that:

$$\begin{aligned}
& (\mathbf{O}_M \mathfrak{Diag}[\pi^\star] \mathbf{Q}_\star \mathfrak{Diag}[\mathbf{O}_M(b, \cdot)] \mathbf{Q}_\star \mathbf{O}_M^\top)(a, c) \\
&= \sum_{(x_1, x_2, x_3) \in \mathcal{X}^3} \mathbf{O}_M(a, x_1) \pi(x_1) \mathbf{Q}_\star(x_1, x_2) \mathbf{O}_M(b, x_2) \mathbf{Q}_\star(x_2, x_3) \mathbf{O}_M(c, x_3), \\
&= \sum_{(x_1, x_2, x_3) \in \mathcal{X}^3} \mathbb{E}[\varphi_a(Y_1) | X_1 = x_1] \mathbb{P}(X_1 = x_1) \mathbb{P}(X_2 = x_2 | X_1 = x_1) \\
&\quad \times \mathbb{E}[\varphi_b(Y_2) | X_2 = x_2] \mathbb{P}(X_3 = x_3 | X_2 = x_2) \mathbb{E}[\varphi_c(Y_3) | X_3 = x_3], \\
&= \mathbb{E}[\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\mathbf{O}_M \mathfrak{Diag}[\pi^\star] \mathbf{Q}_\star^2 \mathbf{O}_M^\top)(a, c) &= \sum_{(x_1, x_2, x_3) \in \mathcal{X}^3} \mathbf{O}_M(a, x_1) \pi(x_1) \mathbf{Q}_\star(x_1, x_2) \mathbf{Q}_\star(x_2, x_3) \mathbf{O}_M(c, x_3), \\
&= \sum_{(x_1, x_2, x_3) \in \mathcal{X}^3} \mathbb{E}[\varphi_a(Y_1) | X_1 = x_1] \mathbb{P}(X_1 = x_1) \mathbb{P}(X_2 = x_2 | X_1 = x_1) \\
&\quad \times \mathbb{P}(X_3 = x_3 | X_2 = x_2) \mathbb{E}[\varphi_c(Y_3) | X_3 = x_3], \\
&= \mathbb{E}[\varphi_a(Y_1) \varphi_c(Y_3)],
\end{aligned}$$

which concludes the proof. \square

Lemma F.2. Let \mathbf{U} be any $(M \times K)$ matrix such that $\mathbf{P}_M \mathbf{U}$ has rank K . Then,

- for all $b \in \{1, \dots, M\}$,

$$\mathbf{B}(b) := (\mathbf{P}_M \mathbf{U})^\dagger \mathbf{M}_M(\cdot, b, \cdot) \mathbf{U} = \mathbf{R} \mathfrak{Diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1},$$

where $\mathbf{R}^{-1} := \mathbf{Q}_\star \mathbf{O}_M^\top \mathbf{U}$ and $(\mathbf{P}_M \mathbf{U})^\dagger := (\mathbf{U}^\top \mathbf{P}_M^\top \mathbf{P}_M \mathbf{U})^{-1} \mathbf{U}^\top \mathbf{P}_M^\top$ denotes the Moore-Penrose pseudoinverse of the matrix $\mathbf{P}_M \mathbf{U}$;

- $\mathbf{U}^\top \mathbf{P}_M \mathbf{U}$ is invertible and, for all $b \in \{1, \dots, M\}$,

$$\mathbf{B}(b) = (\mathbf{U}^\top \mathbf{P}_M \mathbf{U})^{-1} \mathbf{U}^\top \mathbf{M}_M(\cdot, b, \cdot) \mathbf{U} = \mathbf{R} \mathfrak{Diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1}.$$

Proof. Observe that $\mathbf{M}_M(\cdot, b, \cdot) \mathbf{U} = \mathbf{O}_M \mathfrak{Diag}[\pi^\star] \mathbf{Q}_\star \mathfrak{Diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1} = \mathbf{P}_M \mathbf{U} \mathbf{R} \mathfrak{Diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1}$ as claimed. \square

Lemma F.3. Assume that $2\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| < \sigma_K(\mathbf{P}_M)$, then:

(i)

$$\varepsilon_{\mathbf{P}_M} := \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M) - \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|} < 1,$$

(ii)

$$\sigma_K(\hat{\mathbf{P}}_M) \geq \left[\frac{\sigma_K(\mathbf{P}_M) - \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] \sigma_K(\mathbf{P}_M) > \frac{\sigma_K(\mathbf{P}_M)}{2},$$

$$(iii) \quad \sigma_K(\hat{\mathbf{U}}^\top \mathbf{U}) \geq (1 - \varepsilon_{\mathbf{P}_M}^2)^{1/2},$$

$$(iv) \quad \sigma_K(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}}) \geq (1 - \varepsilon_{\mathbf{P}_M}^2) \sigma_K(\mathbf{P}_M),$$

$$(v) \quad \text{for all } \alpha \in \mathbb{R}^K \text{ and for all } v \in \text{Range}(\mathbf{P}_M), \quad \|\hat{\mathbf{U}}\alpha - v\|_2^2 \leq \|\alpha - \hat{\mathbf{U}}^\top v\|_2^2 + \varepsilon_{\mathbf{P}_M}^2 \|v\|_2^2,$$

(vi) if $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ then:

$$\sigma_K(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}) \geq \frac{\sigma_K(\mathbf{P}_M)}{3},$$

(vii)

$$\begin{aligned} \|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1}\| &\leq \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)(1 - \varepsilon_{\mathbf{P}_M}^2)((1 - \varepsilon_{\mathbf{P}_M}^2)\sigma_K(\mathbf{P}_M) - \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}, \\ &\leq 3.2 \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K^2(\mathbf{P}_M)}. \end{aligned}$$

Proof. See Lemma C.1 in [3] for the first five claims. The sixth claim follows from the fourth point and Theorem D.1. The seventh point follows from the fourth claim and Theorem D.2. \square

Control of the observable operator

Claim (iv) in Lemma F.3 and Lemma F.2 ensure that, for all $b \in \{1, \dots, M\}$,

$$\tilde{\mathbf{B}}(b) := (\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}} = \tilde{\mathbf{R}} \mathfrak{Diag}[\mathbf{O}_M(b, \cdot)] \tilde{\mathbf{R}}^{-1},$$

where \mathbf{R}^{-1} may be defined as

$$\tilde{\mathbf{R}}^{-1} := \mathfrak{Diag}[(\|(\mathbf{Q}_\star \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, 1)\|_2, \dots, \|(\mathbf{Q}_\star \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, K)\|_2)] \mathbf{Q}_\star \mathbf{O}_M^\top \hat{\mathbf{U}}.$$

Set $\Lambda := \Theta^\top \hat{\mathbf{U}}^\top \mathbf{O}_M$ and for all $x \in \mathcal{X}$, $\tilde{\mathbf{C}}(x) := \sum_{b=1}^M (\hat{\mathbf{U}} \Theta)(b, x) \tilde{\mathbf{B}}(b) = \tilde{\mathbf{R}} \mathfrak{Diag}[\Lambda(x, \cdot)] \tilde{\mathbf{R}}^{-1}$. Note that $\tilde{\mathbf{R}}$ has unit Euclidean norm columns:

$$\tilde{\mathbf{R}} = (\mathbf{Q}_\star \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1} \mathfrak{Diag}[(\|(\mathbf{Q}_\star \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, 1)\|_2, \dots, \|(\mathbf{Q}_\star \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, K)\|_2)]^{-1},$$

corresponding to unit Euclidean norm eigenvectors of $\tilde{\mathbf{C}}(k)$.

Lemma F.4. Assume that $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$, then, for all $b \in \{1, \dots, M\}$,

$$\|\hat{\mathbf{B}}(b) - \tilde{\mathbf{B}}(b)\| \leq 3.2 \frac{\|\mathbf{M}_M(\cdot, b, \cdot)\|}{\sigma_K(\mathbf{P}_M)} \left[\frac{\|\hat{\mathbf{M}}_M(\cdot, b, \cdot) - \mathbf{M}_M(\cdot, b, \cdot)\|}{\|\mathbf{M}_M(\cdot, b, \cdot)\|} + \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right],$$

and for all $x \in \mathcal{X}$,

$$\|\hat{\mathbf{C}}(x) - \tilde{\mathbf{C}}(x)\| \leq 3.2 \frac{\|\mathbf{M}_M\|_{\infty, 2}}{\sigma_K(\mathbf{P}_M)} \left[\frac{\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty, 2}}{\|\mathbf{M}_M\|_{\infty, 2}} + \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right].$$

Proof. Observe that:

$$\begin{aligned} \|\hat{\mathbf{B}}(b) - \tilde{\mathbf{B}}(b)\| &\leq \|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{M}}_M(\cdot, b, \cdot) \hat{\mathbf{U}} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}}\| \\ &\quad + \|(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}}\|, \\ &\leq \|\hat{\mathbf{U}}^\top (\hat{\mathbf{M}}_M(\cdot, b, \cdot) - \mathbf{M}_M(\cdot, b, \cdot)) \hat{\mathbf{U}}\| \|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1}\| \\ &\quad + \|(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1}\| \|\hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}}\|, \\ &\leq \|\hat{\mathbf{M}}_M(\cdot, b, \cdot) - \mathbf{M}_M(\cdot, b, \cdot)\| \sigma_K^{-1}(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}) \\ &\quad + \|\mathbf{M}_M(\cdot, b, \cdot)\| \|(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1}\|. \end{aligned}$$

By claims (vi) and (vii) of Lemma F.3, $3\sigma_K(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}) \geq \sigma_K(\mathbf{P}_M)$ and $\|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1}\| \leq 3.2 \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K^2(\mathbf{P}_M)}$. Replacing $\mathbf{M}_M(\cdot, b, \cdot)$ by $\sum_{b=1}^M (\hat{\mathbf{U}} \Theta)(b, k) \mathbf{M}_M(\cdot, b, \cdot)$ yields the same result for $\|\hat{\mathbf{C}}(x) - \tilde{\mathbf{C}}(x)\|$. \square

Lemma F.5. Assume that $2\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| < \sigma_K(\mathbf{P}_M)$, then,

(i)

$$\kappa(\tilde{\mathbf{R}}) := \|\tilde{\mathbf{R}}\| \|\tilde{\mathbf{R}}^{-1}\| \leq \kappa^2(\mathbf{Q}_\star \mathbf{O}_M^\top \hat{\mathbf{U}}) \leq \frac{\kappa^2(\mathbf{Q}_\star \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2},$$

(ii)

$$\text{sv}_{\mathbf{C}(1)}(\hat{\mathbf{C}}(1)) \leq \kappa(\tilde{\mathbf{R}}) \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\| \leq \frac{\kappa^2(\mathbf{Q}_\star \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2} \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\|,$$

$$\text{where } \text{sv}_{\mathbf{C}(1)}(\hat{\mathbf{C}}(1)) := \max_{x_1 \in \mathcal{X}} \min_{x_2 \in \mathcal{X}} \left| \hat{\lambda}(1, x_1) - \lambda(1, x_2) \right|.$$

(iii) If in addition,

$$\frac{\kappa^2(\mathbf{Q}_\star \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2} \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\| < \min_{x, x' \in \mathcal{X}} |\Lambda(1, x) - \Lambda(1, x')| / 2,$$

then $\hat{\mathbf{C}}(1)$ has K distinct real eigenvalues and:

$$\text{md}(\mathbf{C}(1), \hat{\mathbf{C}}(1)) \leq \frac{\kappa^2(\mathbf{Q}_\star \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2} \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\|,$$

$$\text{where } \text{md}(\mathbf{C}(1), \hat{\mathbf{C}}(1)) := \min_{\tau \in \mathcal{S}_K} \left\{ \max_{x \in \mathcal{X}} \left| \hat{\Lambda}(1, \tau(x)) - \Lambda(1, x) \right| \right\}.$$

Proof. Observe that \mathbf{U} is an orthonormal basis of range of \mathbf{O}_M . The first point follows from claim (iii) of Lemma F.3. The second point is derived from Theorem D.3 and the first point. The remark following Theorem D.3 proves the last point. \square

Control of the spectra

Lemma F.6. For any $0 < \delta < 1$,

$$\mathbb{P} \left[\forall x, x_1 \neq x_2, |\Lambda(x, x_1) - \Lambda(x, x_2)| \geq \frac{2\delta(1 - \varepsilon_{\mathbf{P}_M}^2)^{1/2}}{\sqrt{e}K^{5/2}(K-1)} \gamma(\mathbf{O}_M) \right] \geq 1 - \delta.$$

Furthermore:

$$\mathbb{P} \left[\|\Lambda\|_\infty \geq \frac{1 + \sqrt{2 \log(K^2/\delta)}}{\sqrt{K}} \|\mathbf{O}_M\|_{2,\infty} \right] \leq \delta.$$

Proof. Observe that:

$$\begin{aligned} \Lambda(x, x_1) - \Lambda(x, x_2) &= \langle \Theta(\cdot, x), (\hat{\mathbf{U}}^\top \mathbf{O}_M)(\cdot, x_1) - (\hat{\mathbf{U}}^\top \mathbf{O}_M)(\cdot, x_2) \rangle \\ &= \langle \Theta(\cdot, x), \hat{\mathbf{U}}^\top (\mathbf{O}_M(\cdot, x_1) - \mathbf{O}_M(\cdot, x_2)) \rangle. \end{aligned}$$

Furthermore, from (iii) in Lemma F.3, we get that:

$$\|\hat{\mathbf{U}}^\top (\mathbf{O}_M(\cdot, x_1) - \mathbf{O}_M(\cdot, x_2))\|_2 \geq (1 - \varepsilon_{\mathbf{P}_M}^2)^{1/2} \|\mathbf{O}_M(\cdot, x_1) - \mathbf{O}_M(\cdot, x_2)\|_2 \geq (1 - \varepsilon_{\mathbf{P}_M}^2)^{1/2} \gamma(\mathbf{O}_M).$$

Similarly, note that:

$$\|\Lambda\|_\infty = \max_{x, x'} |\langle \Theta(\cdot, x), \hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, x') \rangle|,$$

and $\|\hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, x')\|_2 \leq \|\mathbf{O}_M(\cdot, x')\|_2 \leq \|\mathbf{O}_M\|_{2,\infty}$. For sake of readability, we borrow the result of Lemma F.2 and the argument of Lemma C.6 in [3] to conclude. \square

Perturbation of simultaneously diagonalizable matrices

Lemma F.7. If $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ and:

$$8.2K^{5/2}(K-1) \frac{\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{\delta\gamma(\mathbf{O}_M)\sigma_K(\mathbf{P}_M)} \left[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2}\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] < 1, \quad (33)$$

$$43.4K^4(K-1) \frac{\kappa^4(\mathbf{Q}\mathbf{O}_M^\top)}{\delta\gamma(\mathbf{O}_M)\sigma_K(\mathbf{P}_M)} \left[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2}\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] \leq 1, \quad (34)$$

and for all $x, x_1 \neq x_2$,

$$|\Lambda(x, x_1) - \Lambda(x, x_2)| \geq \frac{\sqrt{3}\delta}{\sqrt{e}K^{5/2}(K-1)} \gamma(\mathbf{O}_M),$$

and:

$$\|\Lambda\|_\infty \leq \frac{1 + \sqrt{2 \log(K^2/\delta)}}{\sqrt{K}} \|\mathbf{O}_M\|_{2,\infty},$$

then there exists $\tau \in \mathcal{S}_K$ such that for all $x \in \mathcal{X}$:

$$\begin{aligned} \|\Lambda(\cdot, x) - \hat{\Lambda}(\cdot, \tau(x))\|_\infty &\leq \left[13 \frac{\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{\sigma_K(\mathbf{P}_M)} + 116K^{7/2}(K-1) \left\{ 1 + (2 \log(K^2/\delta))^{1/2} \right\} \right. \\ &\quad \times \left. \frac{\kappa^6(\mathbf{Q}\mathbf{O}_M^\top) \|\mathbf{O}_M\|_{2,\infty}}{\delta \gamma(\mathbf{O}_M) \sigma_K(\mathbf{P}_M)} \right] \times \left[\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2} \|\widehat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right]. \end{aligned}$$

Proof. Note $\varepsilon_{\mathbf{P}_M} \leq 1/2$. Invoke the last part of Claim 4 of Lemma C.4 in [3] with $\gamma_A \leftarrow \frac{\sqrt{3}\delta}{\sqrt{e}K^{\frac{5}{2}}(K-1)}\gamma(\mathbf{O}_M)$, $\kappa(R) \leftarrow \frac{4\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{3}$, $\|\tilde{R}\|_2^2 \leftarrow \frac{4\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{3}$, $\epsilon_A \leftarrow 3.2 \frac{\|\mathbf{M}_M\|_{\infty,2}}{\sigma_K(\mathbf{P}_M)} \left[\frac{\|\widehat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2}}{\|\mathbf{M}_M\|_{\infty,2}} + \frac{\|\widehat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right]$ and $\lambda_{\max} \leftarrow \frac{1 + \sqrt{2 \log(K^2/\delta)}}{\sqrt{K}} \|\mathbf{O}_M\|_{2,\infty}$. Observe that (33) agrees with $\varepsilon_3 < 1/2$ and (34) agrees with $\varepsilon_4 \leq 1/2$. \square

Since Θ^\top is an isometry, observe that:

$$\|\hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, x) - \Theta \hat{\Lambda}(\cdot, \tau(x))\|_2 = \|\Lambda(\cdot, x) - \hat{\Lambda}(\cdot, \tau(x))\|_2 \leq \sqrt{K} \|\Lambda(\cdot, x) - \hat{\Lambda}(\cdot, \tau(x))\|_\infty.$$

Claim (v) in Lemma F.3 (with $\alpha = \Theta \hat{\Lambda}(\cdot, \tau(x))$ and $v = \mathbf{O}_M(\cdot, x)$) give

$$\begin{aligned} \|\mathbf{O}_M(\cdot, x) - \hat{\mathbf{O}}_M(\cdot, \tau(x))\|_2 &\leq \|\hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, x) - \Theta \hat{\Lambda}(\cdot, \tau(x))\|_2 + \frac{3\|\widehat{\mathbf{P}}_M - \mathbf{P}_M\|}{2\sigma_K(\mathbf{P}_M)} \|\mathbf{O}_M(\cdot, x)\|_2 \\ &\leq \sqrt{K} \|\Lambda(\cdot, x) - \hat{\Lambda}(\cdot, \tau(x))\|_\infty + \frac{3\|\widehat{\mathbf{P}}_M - \mathbf{P}_M\|}{2\sigma_K(\mathbf{P}_M)} \|\mathbf{O}_M(\cdot, x)\|_2. \end{aligned}$$

Theorem C.3 follows from Lemma F.7.

References

- [1] G. Alexandrovich and H. Holzmman. Nonparametric identification of hidden Markov models. *arXiv preprint arXiv:1404.4210*, 2014.
- [2] E. S. Allman, C. Matias, and J. A. Rhodes. Identifiability of parameters in latent structure models with many observed variables. *Ann. Statist.*, 37(6A):3099–3132, 12 2009.
- [3] A. Anandkumar, D. Hsu, and S. M. Kakade. A method of moments for mixture models and hidden Markov models. *arXiv preprint arXiv:1203.0683*, 2012.
- [4] J.-P. Baudry, C. Maugis, and B. Michel. Slope heuristics: overview and implementation. *Stat. Comput.*, 22(2):455–470, 2012.
- [5] L.E. Baum, T.P. Petrie, G. Soules, and N. Weiss. A maximization technique occurring in the statistical analysis of probabilistic functions of Markov chains. *Ann. Math. Statist.*, 41:164–171, 1970.
- [6] O. Cappé, É. Moulines, and T. Rydén. *Inference in Hidden Markov Models*. Springer, 2005.
- [7] Y. De Castro, E. Gassiat, and C. Lacour. Minimax adaptative estimation of non-parametric hidden Markov models. *arXiv:1501.04787 [stat.ST]*, 2015.
- [8] P. Del Moral. *Feynman-Kac Formulae. Genealogical and Interacting Particle Systems with Applications*. Springer, 2004.
- [9] P. Del Moral, A. Doucet, and S. Singh. A backward particle interpretation of Feynman-Kac formulae. *ESAIM M2AN*, 44(5):947–975, 2010.
- [10] R. Douc, A. Garivier, É. Moulines, and J. Olsson. Sequential Monte Carlo smoothing for general state space hidden Markov models. *Ann. Appl. Probab.*, 21(6):2109–2145, 2011.

- [11] R. Douc, É. Moulines, and D. Stoffer. *Nonlinear Time Series: Theory, Methods and Applications with R Examples*. Chapman & Hall, 2013.
- [12] A. Doucet, N. De Freitas, and N. Gordon, editors. *Sequential Monte Carlo Methods in Practice*. Springer, New York, 2001.
- [13] A. Doucet, S. Godsill, and C. Andrieu. On sequential Monte Carlo sampling methods for bayesian filtering. *Stat. Comput.*, 10:197–208, 2000.
- [14] C. Dubarry and S. Le Corff. Non-asymptotic deviation inequalities for smoothed additive functionals in nonlinear state-space models. *Bernoulli*, 19(5B):2222–2249, 2013.
- [15] E. Even-Dar, S.M. Kakade, and Y. Mansour. The value of observation for monitoring dynamic systems. *IJCAI*, pages 2474–2479, 2007.
- [16] É. Gassiat, A. Cleyne, and S. Robin. Inference in finite state space non parametric hidden Markov models and applications. *Stat. Comput.*, pages 1–11, 2015.
- [17] S. Godsill, A. Doucet, and M. West. Monte Carlo smoothing for nonlinear time series. *J. Am. Statist. Assoc.*, 50:438–449, 2004.
- [18] D. Hsu, S. M. Kakade, and T. Zhang. A spectral algorithm for learning hidden Markov models. *J. Comput. System Sci.*, 78(5):1460–1480, 2012.
- [19] M. Hürzeler and H.R. Kusch. Monte Carlo approximations for general state-space models. *J. Comput. Graph. Statist.*, 7:175–193, 1998.
- [20] N. Kantas, A. Doucet, S.S. Singh, J. Maciejowski, and N. Chopin. On particle methods for parameter estimation in state-space models. *arXiv:1412.8695v1*, 2014.
- [21] G. Kitagawa. Monte-Carlo filter and smoother for non-Gaussian nonlinear state space models. *J. Comput. Graph. Statist.*, 1:1–25, 1996.
- [22] Y. Meyer. *Wavelets and operators*, volume 37 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. Translated from the 1990 French original by D. H. Salinger.
- [23] J. Olsson and J. Westerborn. Efficient particle-based online smoothing in general hidden markov models: the PaRIS algorithm. *arXiv:1412.7550*, 2014.
- [24] D. Paulin. Concentration inequalities for Markov chains by Marton couplings. *arXiv preprint arXiv:1212.2015v3*, 2014.
- [25] L.R. Rabiner. A tutorial on hidden Markov models and selected applications in speech recognition. *Proc. IEEE*, 77:257–285, 1989.
- [26] G. W. Stewart and J.-G. Sun. *Matrix perturbation theory*. Academic press, 1990.
- [27] V.B. Tadić. Analyticity, convergence, and convergence rate of recursive maximum-likelihood estimation in hidden markov models. *IEEE Transactions on Information Theory*, 56(12), 2010.
- [28] P.-Å. Wedin. Perturbation bounds in connection with singular value decomposition. *BIT Numerical Mathematics*, 12(1):99–111, 1972.